Supersymmetric Noether Currents and Seiberg-Witten Theory

by

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Declaration

This thesis is entirely my own work. No part of it has been previously submitted as an exercise for any degree at any university. This thesis is based on one paper in process of publication that is fully integrated into the body of the thesis.

Alfredo Iorio

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A mio padre e mia madre
To my father and my mother

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General Introduction

"*Credo di essere semplicemente un uomo medio,*

che ha le sue convinzioni profonde,

e che non le baratta per niente al mondo".

A. Gramsci, Lettera dal carcere del 12.XI.1927

The impact of Noether theorem [1] in physics could be the subject of more than one thesis of philosophy of science. The motivations behind it are at the core of the contemporary approach to theoretical physics based on various versions of the symmetry principle. We suggest here to the reader some historical and philosophical references [2] for those who like these topics as well as *hard core* physics.

This thesis has been devoted to the construction of the Noether supercharges for the Seiberg-Witten (SW) model [3]. One of our most important results is the first complete and direct derivation of the SW version of the mass formula [4].

The astonishing results obtained by Seiberg and Witten in their seminal papers are commented in a variety of review papers since their work was published in 1994. Their most exciting achievement is the exact solution of a *quasi*-realistic quantum Yang-Mills model in four dimensions which leads

to the explanation, within the model, of the confinement of electric charge along the lines of the long suspected *dual* Meissner effect. The spin-off's are various and in a wide range of related fields, among others surely there is Supersymmetry (Susy) itself, their model being strongly based on the very special features of N=2 Susy.

Although Susy has been extensively developed this is not the case for Susy Noether currents and charges and this is regrettable because many aspects of Susy theories could be clarified by the currents. A case in point is the SW model where the occurrence of a non-trivial central charge Z is vital. In a nutshell the important features of Z are:

- It allows for SSB of the gauge symmetry within the supersymmetric theory.
- It produces the complete and exact mass spectrum given by[1] $M = |Z| = \sqrt{2}|n_e a + n_m a_D|$.
- It exhibits an explicit SL(2, **Z**) duality symmetry whereas this symmetry is not a symmetry of the theory in the Noether sense.
- It is the most important global piece of information at our disposal, therefore it is vital for the exact solution of the model.

Susy Noether currents present quite serious difficulties due to the following reasons. First Susy is a space-time symmetry therefore the standard procedure to find Noether currents does not give a unique answer. A term, often called *improvement*, has to be added to the term one would obtain for an

[1] n_e and n_m are the electric and magnetic charges respectively and a and a_D are the v.e.v.'s of the scalar field and its dual surviving the Higgs phenomenon in the spontaneously broken phase SU(2) $\rightarrow$ U(1).

internal symmetry. The additional term is not unique, it can be fixed only by requiring the charge to produce the Susy transformations one starts with, and for non trivial theories it is by no means easy to compute. Second the linear realization of Susy involves Lagrange multipliers called dummy-fields, which of course have no canonical conjugate momenta. On the other hand, if dummy-fields are eliminated to produce a standard Lagrangian, then the variations of the fields are no longer linear and the Noether currents are no longer bilinear. A further problem is that the variations of the fields involve space-time derivatives and this happens in a *fermi-boson asymmetric* way (the variations of the fermions involve derivatives of the scalars but not conversely). This implies some double-counting solved only by a correct choice of the current. Besides these problems we also had to deal with an effective Lagrangian. In this case the Lagrangians are not constrained by renormalizability requirements, as it is the case for SW effective Lagrangian. For that theory we deal with terms quartic in the fermions and coefficients of the kinetic terms non-polynomial functions of the scalar field. Because of this, the Noether procedure requires a great deal of care.

We have solved all those problems by implementing a canonical formalism in the different cases under consideration. Firstly we construct the Noether currents for the classical limit of the U(1) sector of the theory. In this case Susy is linearly realized regardless of the dummy fields, no complications arising in the effective case are present and the fields are non-interacting. When the procedure is clearly stated in this case we move to the next level, the effective U(1) sector and we see what is left from the classical case and what is new. Now the currents are very different and, for instance, we cannot use a formula one finds classically to overcome the above mentioned fermi-bose asymmetry in the transformations of the fields. Nevertheless the

constraints imposed by Susy are strong enough to force the effective centre to an identical *form* as the classical one[2] proving the SW conjecture that $Z = n_e a + n_m a_D$. The last step is to consider the SU(2) sector. There we find that the canonical procedure implemented in the U(1) sector does not need any further change and our analysis confirms that U(1) is the only sector that contributes to the centre.

Naturally the future work will be the generalization of our results to any Susy theory, possibly to obtain a *Susy-Noether Theorem*. The task is by no means easy due to the above mentioned problems and other difficulties. For instance, as well known, for ordinary space-time symmetries the energy momentum tensor $T_{\mu\nu}$ can be obtained by embedding the theory in a curved space-time with metric $g_{\mu\nu}$, defining $T_{\mu\nu} = \frac{\delta S}{\delta g_{\mu\nu}}$ and then taking the flat-space limit. In Susy the situation is much more complicated because the embedding has to be in a curved *super*space which only has a quasi-metrical structure.

One may also want to investigate the (non-holomorphic) next-to-leading order term in the superfield expansion of the SW effective Action. The presence of derivatives higher than second spoils the canonical approach and Noether procedure cannot be trivially applied. The interest here is to understand how the lack of canonicity and holomorphy (a crucial ingredient for the solution of SW model) affects the currents and charges, and therefore the whole theory itself. Of course this analysis is somehow more general and it could help to understand how to handle the symmetries of full effective Actions.

[2]Of course this does *not* mean that quantum corrections are not present, as is expected to be the case for N=4, but only that having a *dictionary* we could replace classical quantities by their quantum correspondents with no other changes.

Chapter 1

Noether Theorem and Susy

In this Chapter we want to review the difficulties of the Noether standard procedure in relation to Susy. In the first two Sections we shall state Noether theorem and we shall discuss in particular its application to space-time symmetries and Susy. The aim is to clarify some of the points we found either uncovered or obscure or even wrong in literature. We show a recipe we have found to deal with Susy Noether charges, also in the context of an effective field theory. The Chapter ends with the application of this recipe to a supersymmetric toy model, namely the massive Wess-Zumino model.

1.1 Noether Theorem

Given a theory described by an Action $\mathcal{A} = \int d^4x \mathcal{L}(\Phi_i, \partial\Phi_i)$, where Φ_i are fields of arbitrary spin, we define a *symmetry* of the theory a transformation of the coordinates and/or of the fields that leaves $\mathcal{A}$ invariant *without* the use of the equations of motion for the fields (off-shell). The last requirement is crucial because any transformation leaves $\mathcal{A}$ invariant when the fields obey the equations of motion (on-shell). Noether theorem for classical fields states

that

"*To any continuous symmetry of the Action corresponds a conserved charge*".

The invariance of the Action only ensures the invariance of the Lagrangian density up to a total divergence

$$\delta\mathcal{A} = 0 \Rightarrow \delta\mathcal{L} = \partial_\mu V^\mu \tag{1.1}$$

As we shall see V^μ plays a major role in Susy.

There are different ways to prove this theorem[1], the simplest one is obtained in Quantum Mechanics in Hamiltonian formalism [5]. It consists in the observation that $[H, Q] = 0$, where H is the Hamiltonian and Q the charge, tells us at once that time-conserved charges are symmetries of the theory! The classical derivation of the theorem is based on Lagrangian rather than Hamiltonian formulation. For instance, one could prove the theorem by comparing the variation off-shell to the variation on-shell of the Lagrangian density $\mathcal{L}$. On the one hand, by the above given definition of symmetry, one has that off-shell

$$\delta\mathcal{L} = \partial^\mu V_\mu \tag{1.2}$$

On the other hand the same transformation (and any other transformation) on-shell gives

$$\delta\mathcal{L} = \partial^\mu N_\mu \tag{1.3}$$

where

$$N_\mu \equiv \Pi^i_\mu \delta\Phi_i \quad \text{and} \quad \Pi^i_\mu = \frac{\partial\mathcal{L}}{\partial\partial^\mu\Phi_i} \tag{1.4}$$

Therefore one can write a current given by

$$J_\mu = N_\mu - V_\mu \tag{1.5}$$

[1] We leave to Appendix A a detailed discussion of one proof.

that obeys

$$\partial^\mu J_\mu = (\text{E.L.})_i \delta\Phi^i \tag{1.6}$$

where $(\text{E.L.})_i$ stands for the Euler-Lagrange constraint for the field Φ^i, given by $\partial_\mu \Pi^\mu_i - \frac{\partial \mathcal{L}}{\partial \Phi^i}$. Thus the Noether current is conserved on-shell.

We shall call N_μ the *rigid* current as this is the only contribution to the Noether current when rigid internal symmetries are concerned[2].

The other part V_μ is never zero for space-time symmetries and in general is not unique. As a matter of fact it is obtained from (1.2) thus an improvement term $\partial_\nu W^{[\mu\nu]}$ could be added to it with no effects on the theorem. For ordinary space-time transformations it could be written as[3] [6]

$$V_\mu = -\mathcal{L}\delta x_\mu \tag{1.7}$$

For instance, if one considers the translation symmetry of a scalar field theory, for which $\delta_\mu \phi = \partial_\mu \phi$ and $\delta_\mu x_\nu = \eta_{\mu\nu}$, the Noether current is the *canonical* energy-momentum tensor given by

$$T_{\mu\nu} = \Pi_\mu \partial_\nu \phi - \eta_{\mu\nu}\mathcal{L} \tag{1.8}$$

The reader could wonder about the sign of V_μ entering the canonical expression (1.8). The point is that one may also obtain the energy-momentum tensor $T_{\mu\nu}$ in a slightly different way, namely by not explicitly making use

[2] It is a well known fact that *local* gauge symmetries only fix the form of the interaction but do not introduce new charges. For instance, in Quantum Electrodynamics we have $\mathcal{L}_{QED} = -\frac{1}{4}v_{\mu\nu}v^{\mu\nu} + i\bar{\psi}\gamma_\mu(\partial^\mu - ev^\mu)\psi$, which is U(1) locally invariant. This means that $\delta v_\mu = \partial_\mu \theta$ and $\delta\psi = i\theta\psi$, where θ is the x dependent gauge parameter. From Noether theorem it follows that on-shell $J^{\theta\nu} = \partial_\mu(v^{\mu\nu}\theta)$, therefore $Q^\theta \equiv \int d^3x \vec{\nabla}\cdot(\vec{E}\theta) \to 0$ for $\theta \to 0$ at infinity. Therefore the only conserved charge of this theory is $e\int d^3x \bar{\psi}\gamma^0\psi$

[3] See Appendix A

of the V_μ. This is done by considering x-dependent and x-independent variations of the fields, and identifying the $T_{\mu\nu}$ as the coefficient of δx_ν. This is explained in some details in Appendix A (see in particular Eq.(A.10)).

A different way to produce the $T_{\mu\nu}$ is by embedding the Action $\mathcal{A}$ in a curved space-time, computing its derivative with respect to the metric tensor

$$T_{\mu\nu} = \frac{1}{\sqrt{g}} \frac{\delta \mathcal{A}}{\delta g^{\mu\nu}} \tag{1.9}$$

and taking the flat space-time limit. This gives the *symmetric* (Belinfante) energy-momentum tensor but for instance this $T_{\mu\nu}$ is not improved to give $T^\mu_\mu = 0$, as required by the scale symmetry. Equation (1.9) may also give the improved energy-momentum tensor provided that a suitable extra coupling of the fields to the Ricci scalar is introduced [7].

Even if we do not require any improvement there is another problem with space-time symmetries, namely how to express V_μ in terms of canonical momenta Π^i_μ and transformations $\delta\Phi_i$ [8].

From the previous discussion it is clear that many difficulties arise in the computation of the currents when space-time symmetries are involved. Susy is a very special case of space-time symmetry and we shall see in the next Section that extra complications appear.

1.2 Susy-Noether Theorem

We follow the Weyl notation and the conventions of Wess and Bagger [9], explained in some details in Appendix B. For what follows let us introduce the N *extended* Susy algebra, given by

$$\begin{aligned} [Q^L_\alpha \, , \, \bar{Q}_{M\dot{\alpha}}]_+ &= 2\delta^L_M \sigma^\mu_{\alpha\dot{\alpha}} P_\mu && (1.10) \\ [Q^L_\alpha \, , \, Q^M_\beta]_+ &= \epsilon_{\alpha\beta} Z^{[LM]} && (1.11) \end{aligned}$$

$$[\bar{Q}_{L\dot{\alpha}}\ ,\ \bar{Q}_{M\dot{\beta}}]_+ \quad = \quad \epsilon_{\dot{\alpha}\dot{\beta}} Z^*_{[LM]} \tag{1.12}$$

where $[,]_+$ is the anticommutator, $L, M = 1, ..., N$, $\alpha, \dot{\alpha} = 1, 2$, the Q_α's are the supersymmetry charges, P_μ is the four-momentum and $Z^{[LM]}$ are central terms.

It is beyond the scope of this thesis to discuss Susy in all details. A partial list of references on Susy is [10], [11], [12], [13], [14], [15]. We shall explain some of its nice features in the following Chapters. In particular Chapter 2 is a pedagogical introduction to some of the more advanced applications. What we want to say here is that Susy is the only known way to non trivially combine space-time (Poincarè) and internal symmetries of the S matrix, according to the Haag-Lopuszanski-Sohnius generalization [16] of the Coleman-Mandula theorem [17].

The algebra (1.10)-(1.12) is only the part of the full Susy algebra we shall be interested in. Together with the ordinary Poincarè algebra it is referred to as the Super-Poincarè (SP) algebra[4]. From (1.10)-(1.12) it is evident that Susy is a (special kind!) of space-time symmetry. This can be seen for instance by looking at the r.h.s. of Eq. (1.10) where we find P_μ, the generator of translations[5]. Therefore Susy currents share all the difficulties

[4]The rest of the algebra contains the internal symmetry algebra and the non trivial commutations between the Q_α's and the internal symmetry generators.

[5]The space-time nature of Susy becomes more evident in superspace language. Let us consider N=1 for simplicity. The generic group element of the SP group is given by [18]

$$g = \exp\{a_\mu P^\mu + \epsilon^\alpha Q_\alpha + \bar{\epsilon}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}\} \exp\{\omega_{\mu\nu} L^{\mu\nu}\} \tag{1.13}$$

where $L^{\mu\nu}$ are the Lorentz generators, then the coset space of SP/Lorentz is parameterized by the supercoordinates $z^A \equiv (x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ corresponding to the group element

$$\exp\{x_\mu P^\mu + \theta^\alpha Q_\alpha + \bar{\theta}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}\} \tag{1.14}$$

thus the left action by a $g_o \in$ SP is equivalent to a transformation of the supercoodinates

mentioned in the previous Section with respect to space-time currents. The situation is even more complicated now due to the special nature of this symmetry. Following the same approach described in the last Section, for ordinary space-time symmetries, we shall work with *rigid* Susy, namely we shall take the parameters ϵ_α's to be x-independent. Thus, in our approach, being N_μ the part of the current with no ambiguities, the problem amounts to find a suitable V_μ. Of course, one could also obtain the Susy currents by letting the ϵ^α's become local[6] and then identifying the currents as the coefficients of the $\partial_\mu \epsilon_\alpha$'s after variation of the Action and partial integration

$$\delta_{\rm local}\mathcal{A} = \int d^4x \epsilon^\alpha (\partial_\mu \hat{J}^\mu_\alpha) = \int d^4x (\partial_\mu \epsilon^\alpha) \hat{J}^\mu_\alpha + \text{surface terms} \tag{1.15}$$

(see also the discussion following Eq.(1.8)).

The point is that one wants to produce the right (improved) Susy-Noether charges Q^L_α that correctly generate the Susy transformations of the fields, and this is not straightforward. For instance the charges obtained from the currents $\hat{J}^\mu$ in (1.15) need to be improved [19].

Furthermore, although V_μ could be obtained as related to the second-last term in the superfield expansion [15], [11], this V_μ in general does not correspond to the one required. If one also demands the supercurrent to enter a supermultiplet with the R-current and $T_{\mu\nu}$ we have a V_μ different from the

given by

$$\begin{aligned} x^\mu &\to x^\mu + i\epsilon_o \sigma^\mu \bar{\theta} - i\theta\sigma^\mu \bar{\epsilon}_o + \omega_o^{\mu\nu} x_\nu \\ \theta^\alpha &\to \theta^\alpha + \epsilon_o^\alpha + \frac{1}{4}(\sigma_{\mu\nu}\theta)^\alpha \omega_o^{\mu\nu} \\ \bar{\theta}^{\dot{\alpha}} &\to \bar{\theta}^{\dot{\alpha}} + \bar{\epsilon}_o^{\dot{\alpha}} + \frac{1}{4}(\bar{\theta}\bar{\sigma}_{\mu\nu})^{\dot{\alpha}} \omega_o^{\mu\nu} \end{aligned}$$

[6]This is in the same spirit of what discussed earlier for standard local internal symmetries. But in that case no ambiguities due to improvements arise and the current is once and for all given by the rigid one N^μ.

one obtained from superfield expansion and again we cannot produce the Susy transformations. This problem cannot be cured by a simple analogue of Eq.(1.9), to obtain an improved supercurrent from the Action embedded in a curved superspace because only a quasi-metrical structure is given. Note also that there is no simple analogue of Eq.(1.7).

A second point is that the linear realization of Susy involves bosonic Lagrange multipliers called *dummy fields* to which we cannot associate a conjugate momentum and the standard Noether procedure, based on such conjugates, breaks down. Of course the dummy fields can be eliminated by using their Euler-Lagrange equations but this introduces other ambiguities, unsolved in literature. Namely: *when* to put the dummy fields on-shell, before or after the computation of V_μ? Does that mean that all the fields have to be on-shell? Note that this last point is vital since the definition of symmetry in the first place is based on the variation off-shell of the Action.

Finally, the probably best known feature of Susy is that it transforms fermions into bosons and *vice versa*. It does so by transforming fermions into derivatives of the bosons and bosons into fermions. Therefore the conjugate momenta of the bosons appear in the Susy transformations of the fermions but the contrary is not true in general. This makes even more difficult to express the full current in terms of canonical momenta and transformations.

In a nutshell the difficulties of Susy-Noether currents are

- Susy is a *super*space-time symmetry;
- Susy involves dummy-fields;
- Susy variations involve space-time derivatives in a way not symmetrical with respect to fields of different spin.

For the application in which we shall be more interested in, the SW model [3], the situation is even more complicated due to the following problem that closes the list of difficulties encountered in the computations illustrated later:

- Effective Lagrangians, even non-Susy.

Namely, as we shall see, in SW theory we have to deal with effective Lagrangians and renormalization does not constraint the fermionic terms to be bilinear and the coefficients of the kinetic terms to be constant and in general this is not true. As a matter of fact, the SW effective Lagrangian is quartic in the fermionic fields and has coefficients of the kinetic terms that are non-polynomial functions of the scalar field. Because of this, the Noether procedure requires a great deal of care[7]. For example we shall encounter equal time commutations (Poisson brackets[8]) between fermions and bosons such as

$$\{\psi \ , \ \pi_\phi\} = f(\phi)\psi \quad \text{from} \quad \{\pi_{\bar{\psi}} \ , \ \pi_\phi\} = 0 \tag{1.16}$$

where $f(\phi)$ is a non-polynomial function of the scalar field related to the coefficient of the kinetic terms. This reflects the difficulty of treating Noether currents in a quantum context [6].

All these problems are solved in our analysis and we give here the recipe we have found:

- The Susy-Noether charge that correctly reproduces the Susy transformations is the one obtained from $J^\mu = N^\mu - V^\mu$ where $\delta\mathcal{L} = \partial_\mu V^\mu$

[7] Generally speaking, the Noether procedure has always to be handled with care when applied to quantum theories. On this point see [6].

[8] See Appendix C

and V^{μ} has to be extracted as it is, i.e. no terms like $\partial_{\nu} W^{[\nu\mu]}$ have to be added.

- The variation $\delta\mathcal{L}$ has to be performed off-shell by the definition of symmetry. Nevertheless the dummy fields, and *only* them, automatically are projected on-shell.

- The full current J^{μ} contains terms of the form $\pi_{\psi}\delta\psi$, that generate the fermionic transformations. The *same* term can be written as $\pi_{\phi}\delta\phi + \cdots$ therefore it also generates the bosonic transformations. The situation is more complicated for effective theories.

- The canonical commutation relations are preserved also at the effective level, even if some of the usual assumptions, such as that at equal time all fermions and bosons commute, are incorrect. Noether currents at the effective level do not exhibit the same simple expressions as at the classical level.

Of course a recipe is not a final solution and lot of work has to be done to fully understand the issue of Susy-Nother currents or more generally space-time Noether currents. Nevertheless our work surely is a guideline in this direction and successfully solved the problem of the SW Susy currents that we intended to study.

1.3 Wess-Zumino model

Before starting our journey to the analysis of SW theory, we want to apply the above outlined recipe to the simplest N=1 supersymmetric model where the dummy fields couple to dynamical fields: the Wess-Zumino massive model.

The Lagrangian density and supersymmetric transformations of the fields for this model are given by [9]

$$\mathcal{L} = -\frac{i}{2}\psi\ \partial\!\!\!/\bar{\psi} - \frac{i}{2}\bar{\psi}\ \bar{\partial\!\!\!/}\psi - \partial_\mu A \partial^\mu A^\dagger + FF^\dagger + mAF + mA^\dagger F^\dagger - \frac{m}{2}\psi^2 - \frac{m}{2}\bar{\psi}^2 \tag{1.17}$$

and

$$\delta A = \sqrt{2}\epsilon\psi \qquad \delta A^\dagger = \sqrt{2}\bar{\epsilon}\bar{\psi} \tag{1.18}$$

$$\delta\psi_\alpha = i\sqrt{2}(\sigma^\mu\bar{\epsilon})_\alpha \partial_\mu A + \sqrt{2}\epsilon_\alpha F \quad \delta\bar{\psi}^{\dot{\alpha}} = i\sqrt{2}(\bar{\sigma}^\mu\epsilon)^{\dot{\alpha}}\partial_\mu A^\dagger + \sqrt{2}\bar{\epsilon}^{\dot{\alpha}}F^\dagger \tag{1.19}$$

$$\delta F = i\sqrt{2}\bar{\epsilon}\ \bar{\partial\!\!\!/}\psi \qquad \delta F^\dagger = i\sqrt{2}\epsilon\ \partial\!\!\!/\bar{\psi} \tag{1.20}$$

where A is a complex scalar field, ψ is its Susy fermionic partner in Weyl notation and F is the complex bosonic dummy field.

A note on partial integration in the fermionic sector of (1.17) is now in order. We see that ψ and $\bar{\psi}$ play the double role of *fields* and *momenta* at the same time. It is just a matter of taste to choose Dirac brackets for this second class constrained system [20] or to partially integrate to fix a proper phase-space and implement the canonical Poisson brackets.

If one chooses the canonical Poisson brackets, as we did, then it is only a matter of convenience when to partially integrate the fermions. In fact, even if N^μ and V^μ both change under partial integration, the total current J^μ is *formally* invariant, namely its expression in terms of fields and their derivatives is invariant but the interpretation in terms of momenta and variations of the fields is different. Of course both choices give the same results, therefore one could either start by fixing the proper phase space since the beginning or just do it at the end.

Let us keep (1.17) as it stands, define the following non canonical momenta

$$\pi^\mu_{\psi\alpha} = \frac{i}{2}(\sigma^\mu\bar{\psi})_\alpha \quad \pi^{\mu\dot{\alpha}}_{\bar{\psi}} = \tfrac{i}{2}(\bar{\sigma}^\mu\psi)^{\dot{\alpha}} \tag{1.21}$$

$$\pi^\mu_A = -\partial^\mu A^\dagger \qquad \pi^\mu_{A^\dagger} = -\partial^\mu A \tag{1.22}$$

and use Eq. (1.5) to obtain the supersymmetric current J^μ.

We use the first two ingredients of the recipe to compute V^μ by varying (1.17) off-shell, under the given transformations, obtaining

$$\begin{aligned} V^\mu &= \delta A \pi^\mu_A + \delta A^\dagger \pi^\mu_{A^\dagger} - \delta^A \psi \pi^\mu_\psi - \delta^{A^\dagger} \bar\psi \pi^\mu_{\bar\psi} + \delta^F \psi \pi^\mu_\psi + \delta^{F^\dagger} \bar\psi \pi^\mu_{\bar\psi} \\ & \quad -2\delta^{F_{\rm on}} \psi \pi^\mu_\psi - 2\delta^{F^\dagger_{\rm on}} \bar\psi \pi^\mu_{\bar\psi} \end{aligned} \tag{1.23}$$

where $\delta^X Y$ stands for the part of the variation of Y which contains X (for instance $\delta^F \psi$ stands for $\sqrt{2}\epsilon F$) and $F_{\rm on}$, $F^\dagger_{\rm on}$ are the dummy fields given by their expressions on-shell ($F = -mA^\dagger$, $F^\dagger = -mA$). Note here that we succeeded in finding an expression for V^μ in terms of π^μ's and variations of the fields. Note also that the terms involving $F_{\rm on}$ and $F^\dagger_{\rm on}$ were obtained without any request but they simply came out like that.

Then we write the rigid current

$$N^\mu = \delta A \pi^\mu_A + \delta A^\dagger \pi^\mu_{A^\dagger} + \delta\psi \pi^\mu_\psi + \delta\bar\psi \pi^\mu_{\bar\psi} \tag{1.24}$$

and the full current is given by

$$J^\mu = N^\mu - V^\mu = 2(\delta^{\rm on}\psi \pi^\mu_\psi + \delta^{\rm on}\bar\psi \pi^\mu_{\bar\psi}) \tag{1.25}$$

therefore $J^\mu = 2(N^\mu)^{\rm on}_{\rm fermi}$, with obvious notation. In the bosonic sector N^μ completely cancels out against the correspondent part of V^μ. In the fermionic sector $\delta^F \psi \pi^\mu_\psi$ in N^μ cancels out against the term coming from V^μ, $\delta^A \psi \pi^\mu_\psi$ in N^μ and in V^μ add up and combined with the $2\delta^F_{\rm on} \psi \pi^\mu_\psi$ in V^μ gives $2\delta_{\rm on} \psi \pi^\mu_\psi$ in the full current J^μ. Similarly for $\bar\psi$. This illustrates the third difficulty.

Therefore we conclude that: **a** the dummy fields are on-shell automatically and, if we keep the fermionic non canonical momenta given in (1.21), **b** the full current is given by *twice* the fermionic rigid current $(N^\mu)^{\rm on}_{\rm fermi}$.

The result **a** is the second ingredient of the recipe given above. We shall see in the highly non trivial case of the SW effective Action that this result still holds and it seems to be a general feature of Susy-Noether currents.

The result **b** instead is only valid for simple Lagrangians and it breaks down for less trivial cases. There are two reasons for that curious result: the fictitious double counting of the fermionic degrees of freedom and the third difficulty explained above. Nevertheless, when applicable, Eq.(1.25) remains a labour saving formula. All we have to do is to rewrite J^μ in terms of fields and their derivatives

$$J^\mu = \sqrt{2}(\bar{\psi}\bar{\sigma}^\mu\sigma^\nu\bar{\epsilon}\partial_\nu A + i\epsilon\sigma^\mu\bar{\psi}F_{\rm on} + {\rm h.c.}) \tag{1.26}$$

then choose one partial integration

$$\begin{aligned} J^\mu &= \delta_{\rm on}\psi\pi^{\mu}{}^I_\psi + \sqrt{2}\psi\sigma^\mu\bar{\sigma}^\nu\epsilon\partial_\nu A^\dagger + i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\psi F^\dagger_{\rm on} & (1.27) \\ \text{or} &= \sqrt{2}\bar{\psi}\bar{\sigma}^\mu\sigma^\nu\bar{\epsilon}\partial_\nu A + i\sqrt{2}\epsilon\sigma^\mu\bar{\psi}F_{\rm on} + \delta_{\rm on}\bar{\psi}\pi^{\mu}{}^{II}_{\bar{\psi}} & (1.28) \end{aligned}$$

where $\pi^{\mu}{}^I_\psi = i\sigma^\mu\bar{\psi}$ ($\pi^{\mu}{}^I_{\bar{\psi}} = 0$) and $\pi^{\mu}{}^{II}_{\bar{\psi}} = i\bar{\sigma}^\mu\psi$ ($\pi^{\mu}{}^{II}_{\psi} = 0$) are the canonical momenta obtained by (1.17) conveniently integrated by parts, and perform our computations using canonical Poisson brackets. To integrate by parts in the effective SW theory a greater deal of care is needed due to the fact that the coefficients of the kinetic terms are functions of the scalar field.

Choosing the setting I, for instance, what is left is to check that the charge

$$\mathcal{Q} \equiv \int d^3x J^0(x) = \int d^3x \Big(\delta_{\rm on}\psi\pi^I_\psi + \sqrt{2}\psi\sigma^0\bar{\sigma}^\nu\epsilon\partial_\nu A^\dagger + i\sqrt{2}\bar{\epsilon}\bar{\sigma}^0\psi F^\dagger_{\rm on}\Big) \tag{1.29}$$

correctly generates the transformations. This is a trivial task in this case since the current and the expression of the dummy fields on-shell are very simple and the transformations can be read off immediately from the charge (1.29). We shall see that this is not always the case. It is worthwhile to

notice at this point that to generate the transformations of the scalar field $A^\dagger$ one has to use

$$\delta^A \psi \pi^{\mu}{}^I_\psi = \delta A^\dagger \pi_{A^\dagger} + \sqrt{2}\bar{\psi}\bar{\sigma}^0 \sigma^i \bar{\epsilon} \partial_i A \tag{1.30}$$

Notice also that the transformation of $\bar{\psi}$ is obtained by acting with the charge on the conjugate momentum of ψ: $\{\mathcal{Q}\ ,\ \pi^I_\psi\}_-$.

Chapter 2

SW Theory

In this Chapter we want to introduce the model discovered by Seiberg and Witten in [3], focusing on the aspects we are more interested in. For a complete review we leave the reader to the excellent literature [21], [22], [23], [24], [25], and of course to their beautiful seminal paper.

The solution of this model essentially consists in the computation of a complex function $\mathcal{F}$. This amounts to find *singularities* and *monodromies* and to construct the relative differential equation. We intend to describe this strategy here, by stressing on the vital role of the quantum corrected mass formula, descending from the N=2 Susy.

In the first Section we introduce the model and make clear the mathematical side of the problem. In the second Section we describe in greater detail the physics, showing how the mass formula allows for a very intuitive interpretation of a singularity. In the third Section we introduce electromagnetic (e.m.) duality, again by analyzing the mass formula, and we show how the monodromies around the above mentioned singularities identify a unique $\mathcal{F}$. In the last Section we collect the arguments presented and motivate the

interest in the computation of the central charge of the SW model.

2.1 Introduction

SW model is a N=2 supersymmetric version of a SU(2) Yang-Mills theory in four dimensions.

This is the first and only example of exact solution of a non-trivial four dimensional quantum field theory. The task was achieved by cleverly combining together the following ingredients:

N=2 Susy: holomorphy of the effective Action, non trivial non-renormalization properties, central charge Z;

Spontaneous Symmetry Breaking (SSB): the space of gauge inequivalent vacua in the quantum theory, $\mathcal{M}_q$, exhibits singularities defined in terms of the Higgs vacuum expectation values (v.e.v.'s);

E.M. Duality: electrically charged elementary particles in the asymptotically free sector and magnetically charged topological excitations in the infrared slave sector are exchanged by means of duality.

The N=2 supersymmetric, SU(2) gauged, Wilsonian effective Action[1] in N=2 superfield language is given by [21]

$$\mathcal{A} = \frac{1}{4\pi}\mathrm{Im}\int d^4x d^2\theta d^2\tilde{\theta}\mathcal{F}(\Psi^a\Psi^a) \tag{2.1}$$

where θ and $\tilde{\theta}$ are the grassmanian coordinates of the N=2 superspace[2] and $\Psi^a\Psi^a$, $a = 1, 2, 3$, is the SU(2) gauge Casimir. Ψ^a is the N=2 superfield

[1] The Wilsonian effective Action differs from the standard one particle irreducible effective Action when massive and massless modes are both present. The Wilsonian effective Action allows for the description of the strong coupling regimes in terms of massless (or light) modes only. We shall not enter into details here. For a lucid introduction see [22].

[2] See note on superspace in Chapter 1

that combines a scalar field A^a, a vector field v_μ^a, two Weyl fermions ψ^a and λ^a (and possibly dummy fields) into a single Susy multiplet. Thus all the fields are in the same representation of the gauge group SU(2) as v_μ^a, i.e. the adjoint representation. $\mathcal{F}$ is a holomorphic[3] and analytic[4] function.

The point we want to make here is that the knowledge of the function $\mathcal{F}$, sometimes called *prepotential*, completely determines the theory.

The key idea of Seiberg and Witten is to compute $\mathcal{F}$ by first posing and then solving what mathematicians call a "Riemann-Hilbert (RH) problem[5]" [30], namely: *given as initial data singularities and monodromies, does there exist a Fuchsian system having these data*?

A Fuchsian system is a system of differential equations in the complex domain, given by

$$\frac{df^i(z)}{dz} = A^{ij}(z) f^j(z) \quad i, j = 1, ..., p \tag{2.2}$$

where the $f^i(z)$'s are in general multi-valued complex functions and the matrix $A(z)$ is holomorphic in $S = \mathbf{C} - \{z_1, ..., z_n\}$ and $z_1, ..., z_n$ are poles of $A(z)$ of order at most one. We can naturally associate a group structure to a fundamental system of solutions of (2.2), say[6] $GL(p, \mathbf{C})$. We shall see that

[3] $\mathcal{F}$ is not a function of $\bar{\Psi}$ and this only happens if we stop at the leading order term in the expansion in p_μ of the effective Action. For instance the next-to-leading order term $\mathcal{H}(\Psi, \bar{\Psi})$ is no longer holomorphic [26] [27].

[4] By analytic, we mean that it can have branch cuts, poles etc., but no essential singularities.

[5] In a paper published in 1900 [28] Hilbert presented a list of 23 problems. The statement we are describing here appears as the 21^{st} one in the list. The RH problem seems to be very fruitful in physics. Recently it has been applied to renormalization in Quantum Field Theory [29].

[6] This corresponds to the simple request to have p linearly independent solutions combined together into an invertible $p \times p$ complex matrix, say $F(z) \in GL(p, \mathbf{C})$.

in SW theory this group turns out to be a subgroup of $SL(2, \mathbf{Z})$, namely

$$\Gamma_2 \equiv \Big\{\gamma \in \mathrm{SL}(2,\mathbf{Z}) : \gamma = \mathbf{1} + \begin{pmatrix} l & m \\ n & p \end{pmatrix} l, m, n, p \in \mathbf{Z}\Big\} \tag{2.3}$$

If we now consider the universal covering surface[7] of S, say $\tilde{S}$, we can define maps $\delta : \tilde{S} \to S$. The *monodromy* representation of $GL(p, \mathbf{C})$ is then defined as $M : \delta \to M(\delta) \in GL(p, \mathbf{C})$. More practically the monodromy constant matrices M are obtained by winding around the singularities z_i's of $A(z)$ with loops α_i's in one-to-one correspondence with the z_i's.
Therefore the RH problem consists in finding a system of the type (2.2) starting from the knowledge of the singularities $z_1, ..., z_n$ and the monodromies around them. If at least one of the matrices $M(\alpha_1), ..., M(\alpha_n)$ is diagonalizable then the RH problem has a positive answer[30].
We want to show in the following how these singularities arise in SW model, their physical meaning and the vital role of the central charge Z of the underlying N=2 Susy.

2.2 SSB and mass spectrum

The Action (2.1) is obtained in component fields in the following Chapters and is given by

$$\begin{aligned} \mathcal{A} &= \frac{1}{4\pi}\mathrm{Im}\int d^4x \Big[\mathcal{F}^{ab}\Big(-\frac{1}{4}v^{a\mu\nu}\hat{v}^b_{\mu\nu} - \mathcal{D}_\mu A^a \mathcal{D}^\mu A^{\dagger b} - i\psi^a \not{\mathcal{D}}\bar{\psi}^b - i\lambda^a \not{\mathcal{D}}\bar{\lambda}^b \\ &\quad -\frac{1}{\sqrt{2}}\epsilon^{adc}(A^c\bar{\psi}^b\bar{\lambda}^d + A^{\dagger c}\psi^b\lambda^d) + \frac{1}{2}\epsilon^{acd}\epsilon^{bfg}A^c A^{\dagger d} A^f A^{\dagger g}\Big)\Big] \\ &+ \mathcal{A}_{\mathrm{quantum}} \end{aligned} \tag{2.4}$$

[7] This simply means that we are considering all the Riemann sheets obtained by winding around the singularities $z_1, ..., z_n$. For instance, in the case of a logarithmic function of one complex variable, $\tilde{S}$ represents the infinite copies of the complex plane.

where $\mathcal{F}$ is now a function of the scalar fields only, $\mathcal{F}^{a_1 \cdots a_n} \equiv \partial^n \mathcal{F} / \partial A^{a_1} \cdots \partial A^{a_n}$, $v^{a\mu\nu}$, $\hat{v}^a_{\mu\nu}$ and $\mathcal{D}_\mu$ are the vector field strength, its self-dual projection and the covariant derivative respectively[8].

The Action (2.4) is immediately recognized as (an effective version of) a Georgi-Glashow type of Action. It has: self-coupled gauge fields, topological excitations (instantons and monopoles), gauge fields coupled to matter, a Yukawa potential, and a Higgs potential to spontaneously break the gauge symmetry. The purely quantum term contains third and fourth derivatives of $\mathcal{F}$, vertices with two fermions coupled to the gauge fields and vertices with four fermions. The SU(2) gauge symmetry can be spontaneously broken down to U(1) preserving the N=2 Susy.

This is possible since the Higgs potential $\mathrm{Tr}([\vec{A}, \vec{A}^\dagger])^2$, where $\vec{X} = \frac{1}{2} X^a \sigma^a$ and the σ^a's are the generators of SU(2)[9], admits *flat directions*, i.e. directions in the group that cost no energy. This is the first requirement to spontaneously break SU(2) down to U(1), but preserve Susy at the same time, since the Hamiltonian of a supersymmetric theory is always bounded below. In particular the Higgs potential must be zero on the vacuum[31]. By choosing a direction, say $< 0|A^a|0 >= \delta^{a3} a$, the potential is indeed still zero on the vacuum preserving Susy but spontaneously breaking the gauge symmetry.

We now want to show that the algebraic structure of N=2 Susy indeed allows for a SSB of the gauge symmetry, but only for non-vanishing central charge. The problem is how to handle the jump in the dimension d of the representation of Susy when the Higgs mechanism switches the masses on, but the number of degrees of freedom is left invariant.

[8]These quantities and our SU(2) conventions are all given later in greater detail.

[9]See previous Note.

The irreducible representations of extended Susy are easily found in terms of suitable linear combinations of the supercharges Q^L_α, $L = 1, ..., N$, to obtain creation and annihilation operators acting on a Clifford vacuum [9]. On general grounds one finds that the dimension of the representation of the Clifford algebra corresponding to *massless* states is given by

$$d = 2^N \tag{2.5}$$

while for the *massive* case this number becomes

$$d = 2^{2N} \tag{2.6}$$

As well known, the number at the exponent is the number of the anti-commuting creation and annihilation operators mentioned above[10]. Thus we have a problem if we want to keep Susy in both phases, massless and massive.

The way out was found in [32]. Let us consider the algebra given in (1.10)-(1.12) for N=2, our case. In the rest frame we can write[9]

$$\begin{aligned} [Q^L_\alpha \ , \ (Q^M_\beta)^\dagger]_+ &= 2M \ \delta^L_M \delta^\beta_\alpha & (2.7) \\ [Q^L_\alpha \ , \ Q^M_\beta]_+ &= \epsilon_{\alpha\beta} \ Z^{[LM]} & (2.8) \\ [(Q^L_\alpha)^\dagger \ , \ (Q^M_\beta)^\dagger]_+ &= \epsilon^{\alpha\beta} \ Z^*_{[LM]} & (2.9) \end{aligned}$$

where $L, M = 1, 2$.

By performing a unitary transformation on the Q^L_α we can introduce new charges $\tilde{Q}^L_\alpha = U^L_M Q^M_\alpha$ that obey[11]

$$[\tilde{Q}^L_\alpha \ , \ (\tilde{Q}^M_\beta)^\dagger]_+ = 2M \ \delta^L_M \delta^\beta_\alpha \tag{2.10}$$

[10]There are N (2N) creation and N (2N) annihilation operators in the massless (massive) case.

[11]In this basis $Z^{[LM]}$ is mapped to $\epsilon^{LM} 2|Z|$, where $Z = |Z| e^{i\zeta}$ and $|Z| \geq 0$.

$$[\tilde{Q}^L_\alpha \, , \, \tilde{Q}^M_\beta]_+ \;=\; 2|Z| \, \epsilon_{\alpha\beta}\epsilon^{LM} \tag{2.11}$$

$$[(\tilde{Q}^L_\alpha)^\dagger \, , \, (\tilde{Q}^M_\beta)^\dagger]_+ \;=\; 2|Z| \, \epsilon^{\alpha\beta}\epsilon_{LM} \tag{2.12}$$

where $\epsilon^{LM} = -\epsilon^{ML}$, $\epsilon^{12} = 1 = -\epsilon_{12}$.

We can now define the following annihilation operators

$$a_\alpha \;=\; \frac{1}{\sqrt{2}}(\tilde{Q}^1_\alpha + \epsilon_{\alpha\gamma}(\tilde{Q}^2_\gamma)^\dagger) \tag{2.13}$$

$$b_\alpha \;=\; \frac{1}{\sqrt{2}}(\tilde{Q}^1_\alpha - \epsilon_{\alpha\gamma}(\tilde{Q}^2_\gamma)^\dagger) \tag{2.14}$$

and their conjugates $a^\dagger_\alpha$ and $b^\dagger_\alpha$, in terms of which we can write the algebra as

$$[a_\alpha \, , \, a_\beta]_+ \;=\; [b_\alpha \, , \, b_\beta]_+ = [a_\alpha \, , \, b_\beta]_+ = 0 \tag{2.15}$$

$$[a_\alpha \, , \, a^\dagger_\beta]_+ \;=\; \delta_{\alpha\beta} 2(M + |Z|) \tag{2.16}$$

$$[b_\alpha \, , \, b^\dagger_\beta]_+ \;=\; \delta_{\alpha\beta} 2(M - |Z|) \tag{2.17}$$

For $\alpha = \beta$ the anticommutators (2.16) and (2.17) are never less than zero on any states. Therefore from (2.16) follows $M + |Z| \geq 0$ and from (2.17) follows $M - |Z| \geq 0$. By multiplying these two inequalities together we obtain

$$M \geq |Z| \tag{2.18}$$

Thus, for non-vanishing central charge, the saturation of this inequality, $M = |Z|$, implies that the operators b_α must vanish. This reduces the number of creation and annihilation operators of the Clifford algebra from 4 to 2. Therefore the dimension of the massive representation reduces to the dimension of the massless one: from $2^4 = 16$ to $2^2 = 4$. We have a so-called *short* Susy multiplet.

States that saturate (2.18) are called Bogomolnyi-Prasad-Sommerfield (BPS) states[33]. They are the announced way out from the problem posed by the

Higgs mechanisms: the fields in the massive phase have to belong to the short Susy multiplet, i.e. they have to be BPS states. It is now matter to give physical meaning to the central charge Z arising from the algebra.

We shall concentrate first on the classical case. In [32] the authors considered the classical N=2 supersymmetric Georgi-Glashow Action with gauge group O(3) spontaneously broken down to U(1) and its supercharges. In Chapter 4 we shall compute the quantum central charge for the SU(2) Action (2.4), for the moment let us just write down the classical limit of it that gives back the result obtained in [32]

$$Z = i\sqrt{2}\int d^2\vec{\Sigma}\cdot(\vec{\Pi}^a A^a + \frac{1}{4\pi}\vec{B}^a A_D^a) \qquad a = 1,2,3 \tag{2.19}$$

where $d^2\vec{\Sigma}$ is the measure on the sphere at spatial infinity S^2_∞, the A^a's are the scalar fields, the $\vec{B}^a$'s are the magnetic fields, $\vec{\Pi}^a$ is the conjugate momentum of the vector field $\vec{v}^a$ and $A_D^a \equiv \tau A^a$ where[12]

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2} \tag{2.20}$$

is the classical *complex* coupling constant, g is the SU(2) coupling constant and θ the CP violating vacuum angle[35]. In the classical case A_D^a is merely a formal quantity with no precise physical meaning. On the contrary, in the low-energy sector of the quantum theory, it becomes the *e.m. dual* of the scalar field.

In the unbroken phase $Z = 0$, but, as well known, in the broken phase this theory admits 't Hooft-Polyakov monopole solutions [36]. In this phase the scalar fields (and the vector potentials) tend to their vacuum value $A^a \sim a\frac{r^a}{r}$ ($v^{ai} \sim \epsilon^{iab}\frac{r^b}{r^2}$, $v^{a0} = 0$), where $a \in \mathbf{C}$, as $r \to \infty$. This behavior gives rise

[12]The θ contribution to the complex coupling τ was discovered by Witten in [34] shortly after.

to a magnetic charge. By performing a SU(2) gauge transformation on this radially symmetric ("hedgehog") solution we can align $< 0|A^a|0 >$ along one direction (the Coulomb branch), say $< 0|A^a|0 >= \delta^{a3} a$, and the 't Hooft-Polyakov monopole becomes a U(1) Dirac-type monopole [35], [22], [23].

In this spirit we can define the electric and magnetic charges as [32]

$$q_e \equiv \frac{1}{a} \int d^2\vec{\Sigma} \cdot \vec{\Pi}^3 A^3 \tag{2.21}$$

$$q_m \equiv \frac{1}{a} \int d^2\vec{\Sigma} \cdot \frac{1}{4\pi} \vec{B}^3 A^3 = \frac{1}{a_D} \int d^2\vec{\Sigma} \cdot \frac{1}{4\pi} \vec{B}^3 A^3_D \tag{2.22}$$

where $a_D = \tau a$ and only the U(1) fields remaining massless after SSB appear. These quantities are quantized, since[13] $q_m \in \pi_1(U(1)) \sim \pi_2(SU(2)) \sim \mathbf{Z}$ and q_e is quantized due to Dirac quantization of the electric charge in presence of a magnetic charge[22].

Thus, after SSB, the central charge becomes

$$Z = i\sqrt{2}(n_e a + n_m a_D) \tag{2.23}$$

The mass spectrum of the theory is then given by

$$M = \sqrt{2}|n_e a + n_m a_D| \tag{2.24}$$

We shall call this formula the Montonen-Olive mass formula[14]. It is now crucial to notice that this formula holds for the whole spectrum consisting

[13] We say that the 't Hooft-Polyakov magnetic charge is the winding number of the map $SU(2) \sim S^2 \to S^2_\infty$, that identifies the homotopy class of the map. By considering the maps $U(1) \sim S^1 \to S^1_\infty$, where S^1_∞ is the equator of S^2_∞, it is clear that a similar comment holds for the U(1) Dirac type magnetic charge. It turns out that the two homotopy groups, $\pi_2(S^2)$ and $\pi_1(S^1)$, are isomorphic to $\mathbf{Z}$. For an enjoyable and pedagogical introduction to topological objects and their role in physics I recommend [37].

[14] In our short-cut to write down the classical version of the mass formula, we did not follow the chronological order of the various discoveries that led to it.

of *elementary* particles, two W bosons and two fermions[15], and *topological* excitations, monopoles and dyons. For instance the mass of the W bosons and the two fermions can be obtained by setting $n_e = \pm 1$ and $n_m = 0$, which gives $m_W = m_{\text{fermi}} = \sqrt{2}|a|$, whereas the mass of a monopole ($n_e = 0$ and $n_m = \pm 1$) amounts to $m_{\text{mon.}} = \sqrt{2}|a_D|$. This establishes a democracy between particles and topological excitations that becomes more clear when e.m. duality is implemented.

After this long preparation we are now in the position to introduce the most important tool to reduce the solution of SW model to that of a RH problem in complex analysis: the singularities.

Since the Higgs v.e.v. $a \in \mathbf{C}$, we can think of $\mathbf{C}$ as the space of gauge inequivalent vacua, namely to a, a', with $a \neq a'$, correspond two vacua not related by a SU(2) gauge transformation (but only by a transformation in the little group U(1)). To be more precise we have to introduce the SU(2)

First Bogomolnyi, Prasad and Sommerfield [33] showed that, for a theory admitting monopole solutions, the formula

$$M = a(q_e^2 + q_m^2)^{1/2} \tag{2.25}$$

holds classically for monopoles and dyons (topological excitations carrying electric and magnetic charge). Then Montonen and Olive [38] showed that it is true classically for all the states, elementary particles included. Finally Witten and Olive [32] obtained it, again classically, from the N=2 Susy.

The formula (2.25) can be written in the following form

$$M = |ag(n_e + \tau_0 n_m)| \tag{2.26}$$

where $q_e \equiv g n_e$, $q_m = (-4\pi/g) n_m$ and $\tau_0 \equiv i4\pi/g^2$. This is the formula we are showing here, provided $ag \to a$ and τ_0 is *improved* to τ.

[15] We work with Weyl (chiral) components ψ^a and λ^a, whose masses are generated by the Yukawa potential in (2.4).

invariant parameter

$$u(a) = \mathrm{Tr} < 0|\vec{A}^2|0 >= \frac{1}{2}a^2 \tag{2.27}$$

to get rid of the ambiguity due to the discrete Weyl group of SU(2), which still acts as $a \to -a$ within the Cartan subalgebra. This is now a good coordinate on the complex manifold of gauge inequivalent vacua. We shall call this manifold $\mathcal{M}$, for *moduli* space.
Eventually we can define a singularity of $\mathcal{M}$ as *a value of* u *at which some of the particles of the spectrum, either elementary or topological, become massless*. Classically there is only one of such values, namely $u = 0$ where the SU(2) gauge symmetry is fully restored and $\mathcal{M}$ looses its meaning. It is worthwhile to notice that the classical moduli space is merely a tool to introduce the idea of a singularity, since the running of the coupling is a purely quantum effect, therefore there is no physical reason to vary u classically. Nevertheless the crucial point is to keep the idea of a singularity of $\mathcal{M}$ as a point where *some* particles become massless.

The big step is to go to the quantum theory (2.4) where non trivial renormalization leads to a non vanishing beta function. The running of the effective coupling $g_{\text{eff}}(\mu)$, where μ is the renormalization scale, presented in Figure 2.1, explains why the physics changes dramatically from the high energy regime to the low energy one. In fact at low energies the coupling is expected to become strong and we cannot make reliable predictions based on perturbative analysis as at high energies where the coupling is weak. The masses of the elementary fields in SU(2)/U(1) become big in the low energy sector and the effective theory can be described all in terms of the massless U(1) fields (the heavy fields can be integrated out form the effective Action).

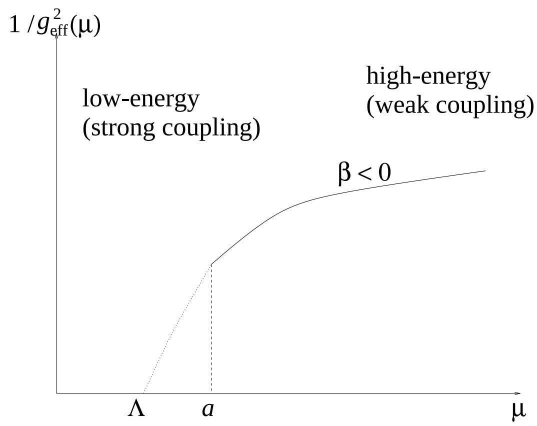

Figure 2.1: Running of the effective coupling. a is the Higgs v.e.v. and Λ is the dynamically generated mass scale at which the W bosons and two fermions are expected to become infinitely heavy.

We can replace μ in $\tau_{\rm eff}(\mu) = \frac{\theta_{\rm eff}(\mu)}{2\pi} + i\frac{4\pi}{g^2_{\rm eff}(\mu)}$ by the Higgs v.e.v.. Thus now is a (therefore u) that varies and $\tau_{\rm eff}(a)$ becomes a field-dependent coupling as often happens in effective theories. Therefore in the quantum theory we can define a proper moduli space $\mathcal{M}_q$.

What happens to the mass formula (2.24)? Seiberg and Witten conjectured that, due to the preservation of the N=2 Susy, the formula is *formally* unchanged: quantum corrections play a major role since now [3]

$$a_D^{\rm class} = \tau_{\rm class} a \to a_D^{\rm eff} \equiv \frac{\partial \mathcal{F}(a)}{\partial a} \tag{2.28}$$

where $\mathcal{F}(a)$ is the prepotential in the low energy sector, evaluated at a (see more on this in the next Section), but no other changes are expected. Therefore the quantum improvement of the Montonen-Olive mass formula (2.24) is simply given by

$$M = \sqrt{2}|n_e a + n_m \mathcal{F}'(a)| \tag{2.29}$$

This statement is vital for the whole theory. Nevertheless no direct proof

from the N=2 Susy algebra was presented. In the following Chapters we shall dedicate most of our attention to this point.

The vital importance of the mass formula is immediately seen if one wants to define the singularities of the quantum theory in the spirit outlined above. In fact Seiberg and Witten conjectured that, in the quantum theory, the singularity at $u = 0$ splits into $u = \pm\Lambda^2$, where monopoles and dyons, and *not* W bosons and fermions (as in the classical case) are supposed to become massless. This makes sense if the W bosons in the low energy sector can decay into a monopole + dyon pair. Since all the states are BPS, one can show [25] that the mass formula (2.29) indeed allows for this decay. Thus, if some particles have to become massless in the low energy sector, these cannot be the W bosons, whose mass is frozen at low energies, but only the topological excitations. Of course this is only a sufficient but not necessary condition for this to happen. Furthermore one should explain why only two singularities and why at $\pm\Lambda^2$ and there is no rigorous proof of these points.

In Figure 2.2 we present a pictorial summary of this Section. We see from the picture that also a third singular point appears at $u = \infty$. We could say that, due to the asymptotic freedom, at that point all the elementary particles become massless. As we shall see in the following Section, this point is somehow on a different footing respect to the other two.

In the following we shall remove the suffix "eff" from the effective quantities, since their field dependence clearly identifies them.

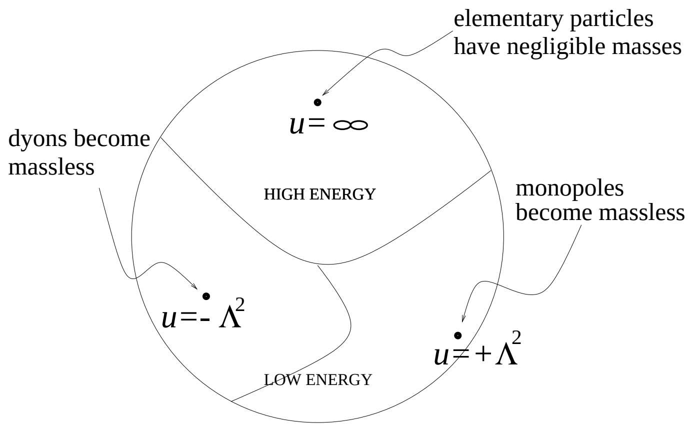

Figure 2.2: The quantum moduli space $\mathcal{M}_q$. The singularities and the different corresponding *phases* are shown.

2.3 Duality and the solution of the model

It is now matter to associate these singularities to the function $\mathcal{F}$ we are looking for and determine the monodromies around them.

For large values of a ($a >> \Lambda$) the theory (2.4) is weakly coupled, thus a perturbative computation to evaluate $\mathcal{F}$ leads to a reliable result. This computation was performed in [39] and it turns out that

$$\mathcal{F}(a) = \frac{1}{2}\tau a^2 + a^2 \frac{i\hbar}{2\pi} \ln(\frac{a^2}{\Lambda^2}) + a^2 \sum_{k=1}^{\infty} c_k \frac{\Lambda^{4k}}{a^{4k}} \tag{2.30}$$

where the function is parameterized by the Higgs v.e.v. a (i.e. evaluated on the vacuum). The first two terms are the perturbative contributions: tree level and one loop terms respectively[16], and the last term is the non-perturbative instanton contribution. From this expression we see that the classical limit consists in the substitution $\mathcal{F}(a) \to \frac{1}{2}\tau a^2$.

[16]For N=2 Susy these are the only two contributions to the perturbative $\mathcal{F}$ (non-renormalization) whereas for N=4 the tree level (classical) term is enough (super-renormalization).

$\mathcal{F}(a)$ is well defined only in the region of $\mathcal{M}_q$ near $u = \infty$ since the instanton sum converges there. If we try to globally extend $\mathcal{F}(a)$ to the whole $\mathcal{M}_q$ this is not longer the case. This can be seen from another perspective. If one requires the mass formula (2.29) to hold on the whole $\mathcal{M}_q$, since at $u = 0$ there are no singularities, $Z|_{u=0} = i\sqrt{2}(a(u)n_e + a_D(u)n_m)|_{u=0} \neq 0$. If we use the relation (2.27) to write $a = \sqrt{2u}$, we expect the elementary particles ($n_e \neq 0$ and $n_m = 0$) to become massless, but this implies $Z|_{u=0} = i\sqrt{2}a(u)|_{u=0} = 0$, which contradicts the hypothesis. Therefore $u = 0 \not\Rightarrow a = 0$ and a is not a good coordinate to evaluate $\mathcal{F}$ in the low-energy sector. We learn here that the functions $a(u)$ and $a_D(u)$ are very different in the three sectors of $\mathcal{M}_q$. All these are clear signals that we need different local descriptions in the weak coupling and strong coupling phases of the quantum theory.

There is a peculiar symmetry, well known in physics, that exchanges weak and strong coupling regimes: $G \to 1/G$, where G indicates a generic coupling. This symmetry is called *duality*[17] and is the way out of our dilemma. Well known examples are certain two-dimensional theories, where duality may exchange different phases of the same theory, as for the Ising model [23], or map solutions of a theory into solutions of a different theory, as for the bosonic Sine-Gordon and fermionic Thirring models [40]. In the latter case duality exchanges the solitonic solutions of the Sine-Gordon model with the elementary particles of the Thirring model.
As explained above this is not a symmetry in the Noether sense, but rather a transformation that connects different phases. To see how this applies to

[17] This is referred to as S duality. Shortly we shall see that in SW theory this duality is represented by only one of the generators of the whole duality group $SL(2, \mathbf{Z})$, the other one corresponding to the T duality [3].

our problem let us look again at the central charge Z

$$Z = i\sqrt{2}(n_e a + n_m a_D) = i\sqrt{2}(n_m, n_e) \begin{pmatrix} a_D \\ a \end{pmatrix} \tag{2.31}$$

If we act with $S^{-1} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ on the row vector (n_m, n_e), we exchange electric charge with magnetic charge and *vice-versa*. This is the e.m. duality transformation: it maps electrically charged elementary particles to magnetically charged collective excitations, giving meaning to the democracy announced above between all the BPS states. In SW theory this is an exact symmetry of the low energy Action, as well explained in [21] and it corresponds to the mapping

$$\tau(a) \to -\frac{1}{\tau_D(a_D)} \tag{2.32}$$

where $\tau(a)$ is the effective coupling introduced in the last Section, and τ_D its dual. Thus by means of this transformation we map the strong coupling regime to the low coupling one and *vice-versa*.

The mass of all the particles, regardless to which phase of $\mathcal{M}_q$ one considers, has to be given by the mass formula (2.29). Therefore to S^{-1} acting on (n_m, n_e) it has to correspond S acting on the column vector, namely

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to S \begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} a \\ -a_D \end{pmatrix} \tag{2.33}$$

so that Z is left invariant. Thus the S duality invariance of Z suggests that the good parameter for $\mathcal{F}$ (or better, its dual $\mathcal{F}_D$) near $u = 0$ is a_D rather than a. As already noted the functions $a_D(u)$ and $a(u)$ are now different from the ones obtained near $u = \infty$, and the task is to find them. The mass formula is actually invariant under the full group $SL(2, \mathbf{Z})$ of 2×2

unimodular matrices with integer entries, generated by

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \tag{2.34}$$

where $b \in \mathbf{Z}$.

We now have to make this symmetry compatible with the singularities by considering the monodromies of $a_D(u)$ and $a(u)$ around them. This will restrict the group of dualities to a subgroup of SL(2, $\mathbf{Z}$) containing the monodromies.

The monodromy at $u = \infty$ can be easily computed, since here we can trust the perturbative expansion (2.30) and we have $a = \sqrt{2u}$ and $a_D(u) \sim i\frac{\hbar}{\pi}\sqrt{2u}\Big(\ln(\frac{2u}{\Lambda^2}) + 1\Big)$. By winding around $u = \infty$, the branch point of the logarithm, we obtain

$$\begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} \rightarrow \begin{pmatrix} a_D(e^{i2\pi}u) \\ a(e^{i2\pi}u) \end{pmatrix} = \begin{pmatrix} -a_D(u) + 2a(u) \\ -a(u) \end{pmatrix} \tag{2.35}$$

or $\begin{pmatrix} a_D \\ a \end{pmatrix} \rightarrow M_\infty \begin{pmatrix} a_D \\ a \end{pmatrix}$ where

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \tag{2.36}$$

This matrix is diagonalizable, therefore the RH problem has a positive solution [30]. We are on the right track!

To find the other two monodromies we require the state of vanishing mass responsible for the singularity to be invariant under the monodromy transformation:

$$(n_m, n_e)M_{(n_m,n_e)} = (n_m, n_e) \tag{2.37}$$

This simply means that, even if SL(2, $\mathbf{Z}$) maps particles of one phase to particles of another phase, once we arrive at a singularity the monodromy does not change this state into another state. From this it is easy to check that the form of the monodromies around the other two points $\pm\Lambda^2$ has to be

$$M_{(n_m,n_e)} = \begin{pmatrix} 1+2n_m n_e & 2n_e^2 \\ -2n_m^2 & 1-2n_m n_e \end{pmatrix} \tag{2.38}$$

Note that M_∞ is not of this form.

The global consistency conditions on how to patch together the local data is simply given by[18]

$$M_{+\Lambda^2} \cdot M_{-\Lambda^2} = M_\infty \tag{2.39}$$

which follows from the fact that the loops around $\pm\Lambda^2$ can be smoothly pull around the Riemann sphere to give the loop at infinity. By using the expression (2.38) we can obtain the solution of this equation given by

$$M_{+\Lambda^2} = M_{(1,0)} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \quad M_{-\Lambda^2} = M_{(1,1)} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \tag{2.40}$$

and we see that the particles becoming massless are indeed monopoles and dyons as conjectured.

The monodromy matrices generate the subgroup Γ_2 of the full duality group $SL(2,\mathbf{Z})$ given in (2.3).

[18] In the Ising model the *gluing* of the different local data consists in the identification of a self-dual point $K = K^*$, where $K = J/k_B T << 1$ is the coupling at high temperature T and $K^* >> 1$ is the coupling at low temperature given by $\sinh(2K^*) = (\sinh(2K))^{-1}$, J is the strength of the interaction between nearest neighbors and k_B the Boltzman constant. This determines exactly the critical temperature of the phase transition T_c given by $\sinh(2J/k_B T_c) = 1$ [23].

We have now all the ingredients and we can write down the announced Fuchsian equation[19]

$$\frac{d^2 f(z)}{dz^2} = A(z) f(z) \tag{2.41}$$

where [21]

$$A(z) = -\frac{1}{4}\Big[\frac{1-\lambda_1^2}{(z+1)^2} + \frac{1-\lambda_2^2}{(z-1)^2} - \frac{1-\lambda_1^2-\lambda_2^2+\lambda_3^2}{(z+1)(z-1)}\Big] \tag{2.42}$$

$z \equiv u/\Lambda^2$ and $A(z)$ exhibits the described singularities at $z = \pm 1$ and $z = \infty$. Seiberg and Witten have found that the coefficients are $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 0$, thus

$$A(z) = -\frac{1}{4}\frac{1}{(z+1)(z-1)} \tag{2.43}$$

The two solutions of (2.41) with $A(z)$ in (2.43), are given in terms of hypergeometric functions. By using their integral representation one finally obtains [3], [21]

$$f_1(z) \equiv a_D(z) = \frac{\sqrt{2}}{\pi}\int_1^z dx \frac{\sqrt{x-z}}{\sqrt{x^2-1}} \tag{2.44}$$

$$f_2(z) \equiv a(z) = \frac{\sqrt{2}}{\pi}\int_{-1}^1 dx \frac{\sqrt{x-z}}{\sqrt{x^2-1}} \tag{2.45}$$

We can invert the second equation to obtain $z(a)$ then substitute this into $a_D(z)$ to obtain $a_D(a) = \partial_a \mathcal{F}(a)$. Integrating with respect to a yields to $\mathcal{F}(a)$. Thus the theory is solved!

As noted above this expression of $\mathcal{F}(a)$ is not globally valid on $\mathcal{M}_q$, but only near $u = \infty$. For the other two regions one has [25] $\mathcal{F}_D(a_D)$ near $u = +\Lambda^2$ and $\mathcal{F}_D(a - 2a_D)$ near $u = -\Lambda^2$. The unicity of this solution was proved in [41].

[19] This is a second order differential equation therefore it is equivalent to a Fuchsian system (2.2) with $p = 2$. Note also that the poles of $A(z)$ become second order.

Let us conclude this quick *tour de force* on SW model by saying that this theory is surely an exciting laboratory to study the behavior of gauge theories at the quantum core. Nevertheless it is strongly based on N=2 Susy and, at the present status of the experiments, Nature does not even show any clear evidence for N=1 Susy[20]!

2.4 The computation of the effective Z

As we hope is clear from the previous Sections, the mass formula

$$M = |Z| = \sqrt{2}|n_e a + n_m a_D| \tag{2.46}$$

plays a major role in SW model. Let us stress here again that the knowledge of the central charge Z amounts to the knowledge of the mass formula.
In a nutshell the important features of Z are:

- It allows for SSB of the gauge symmetry within the supersymmetric theory.
- It gives the complete and exact mass spectrum. Namely it fixes the masses for the elementary particles as well as the collective excitations.
- It exhibits an explicit SL(2, **Z**) duality symmetry whereas this symmetry is not a symmetry of the theory in the Noether sense.
- In the quantum theory it is the most important global piece of information at our disposal on $\mathcal{M}_q$. Therefore it is vital for the exact solution of the model.

[20] There is an intense search for N=1 superparticles in the accelerators. For instance, the next generation of linear colliders will run at ranges of final energy 2 TeV [42], where signals of N=1 Susy are expected.

It is then not surprising that, following the paper of Seiberg and Witten, there has been a big interest in the computation of the mass formula in the quantum case. As a matter of fact, in their paper there is no direct proof of this formula but only a check that the bosonic terms of the SU(2) high energy effective Hamiltonian for a magnetic monopole admit a BPS lower bound given by $\sqrt{2}|\mathcal{F}'(a)|$ [3].

A similar type of BPS computation, only slightly more general, has been performed in [43]. There the authors considered again the SU(2) high energy effective Hamiltonian but this time for a dyon, namely also the electric field contribution was considered. By Legendre transforming the Lagrangian given in (2.4), one sees that the bosonic terms of the Hamiltonian, in the gauge $\mathcal{D}_0 A^a = \mathcal{D}_0 A^{\dagger a} = 0$ and for vanishing Higgs potential, are given by

$$H = \frac{1}{8\pi}\mathrm{Im}\int d^3x \mathcal{F}^{ab}(E_i^a E_i^b + B_i^a B_i^b + 2\mathcal{D}_i A^a \mathcal{D}_i A_i^{\dagger b}) \tag{2.47}$$

where the electric and magnetic fields are defined as $E_i^a = v_{0i}^a$ and $B_i^a = \frac{1}{2}\epsilon_{0ijk}v^{jk}$, respectively, $a, b = 1, 2, 3$ are the SU(2) indices and $i, j, k = 1, 2, 3$ are the spatial Minkowski indices, $\mathcal{F}$ is a function of the scalar fields only. By using the Bogomolnyi *trick* to complete the square one can write this part of the Hamiltonian as the sum of two contributions, one dynamical and one topological: $H = H_0 + H_{\text{top}}$. Explicitly we have

$$H_0 = \frac{1}{8\pi}\mathrm{Im}\int d^3x \mathcal{F}^{ab}(B_i^a + iE_i^a + \sqrt{2}\mathcal{D}_i A^a)(B_i^a - iE_i^a + \sqrt{2}\mathcal{D}_i A^{\dagger a}) \tag{2.48}$$

$$H_{\text{top}} = -\frac{\sqrt{2}}{8\pi}\mathrm{Im}\int d^3x \mathcal{F}^{ab}\Big((B_i^a - iE_i^a)\mathcal{D}_i A^a + (B_i^a + iE_i^a)\mathcal{D}_i A^{\dagger a}\Big) \tag{2.49}$$

Of course the topological term (2.49) is the lower bound for H. The inequality $H \geq H_{\text{top}}$ is saturated when the configurations of the fields satisfy the

BPS equations [33] (BPS configurations)

$$B_i^a + iE_i^a + \sqrt{2}\mathcal{D}_i A^a = 0 \tag{2.50}$$

Note that these equations hold in the classical case with no changes.
The authors in [43] found that

$$H_{\text{top}} = \sqrt{2}|n_e a + n_m \mathcal{F}'(a)| \tag{2.51}$$

therefore they identified the r.h.s. of this equation with the modulus of the central charge $|Z|$.
This computation is rather unsatisfactory since it only considers the bosonic contributions to $|Z|$ and, due to Susy, one has to expect fermionic terms to play a role. Furthermore it is too an indirect computation of the central charge. The complete and direct computation has to involve the Noether supercharges constructed from the Lagrangian (2.4). As discussed earlier, Witten and Olive [32] have done that in the classical case. But for the effective case a direct and complete derivation is in order. We shall dedicate most of our attention to this point in the rest of this thesis.
Firstly we shall concentrate on the U(1) low energy sector of the theory, since the U(1) massless fields are supposed to be the only ones contributing to the central charge. As a warming-up we shall re-obtain the classical results of Witten and Olive [32]. Then we shall move to the U(1) effective case to compare this case with the classical one and give the first direct and complete derivation of the mass formula [4].
The SU(2) high energy sector is analyzed in the last Chapter. The main interest there is to check the role of the massive fields in SU(2)/U(1) with respect to the central charge.
The other interest, not less important, is the application of the Noether

procedure to find effective supercurrents and charges, as explained in some details in the first Chapter.

The kind of computation we are considering also seems to follow from a geometric analysis of the N=2 vector multiplet in [44], where, however, the authors' aim there is completely different, the fermionic contribution is not present and there is no mention of Noether charges. On the other hand an independent complete computation [19] was performed while we were working on the SU(2) sector. We shall present our independent results for the SU(2) sector referring to this computation as a double-checking of our formulae.

Chapter 3

SW U(1) Low Energy Sector

In this Chapter we shall construct the Noether Susy currents and charges for the SW U(1) low energy Action. The second Section is dedicated to the classical case, where we shall set up a canonical formalism, necessary for the implementation of the Noether procedure for constructing the currents and the charges, as explained in Chapter 1. In this Section the classical central charge of the N=2 Susy is re-obtained. The result is in agreement with [32]. In the third Section we shall deal with the highly non trivial case of the quantum corrected theory. We shall show that the canonical setting still survives, but many delicate issues have to be handled with care. We shall compute the non trivial contributions to the full currents, which we christened V_μ in Chapter 1. Then, after having tested these results by obtaining the Susy transformations from the Susy charges, we shall compute the effective central charge Z. This computation is the first complete and direct proof of the correctness of SW mass formula and it appears in [4].

3.1 Introduction

There exist two massless N=2 Susy multiplets with maximal helicity 1 or less: the vector multiplet and the scalar multiplet [14], [15]. We are interested in the vector multiplet Ψ, also referred to as the N=2 Yang-Mills multiplet, for the moment in its Abelian formulation. Its spin content is $(1, \frac{1}{2}, \frac{1}{2}, 0, 0)$ and, in terms of physical fields, it can accommodate 1 vector field v_μ, 2 Weyl fermions ψ and λ, one complex scalar A. The N=2 vector multiplet can be arranged into two N=1 multiplets, the vector (or Yang-Mills) multiplet W and the scalar multiplet Φ, related by R-symmetry: $\psi \leftrightarrow -\lambda$, $E^\dagger \leftrightarrow E$ and $v^\mu \to -v^\mu$ (charge conjugation). In terms of component fields the N=1 multiplets are given by

$$W = (\lambda_\alpha,\ v_\mu,\ D) \quad \text{and} \quad \Phi = (A,\ \psi_\alpha,\ E) \tag{3.1}$$

where E and D are the (bosonic) dummy fields[1]. Note that W is a real multiplet and Φ is complex. This means that v_μ and D are real, and W contains also $\bar\lambda$, as can be seen by the Susy transformations given below. Of course the complex conjugate of Φ is given by $\Phi^\dagger = (E^\dagger,\ \bar\psi_{\dot\alpha},\ A^\dagger)$.

The N=2 Susy transformations of these fields are well known [14], [15], [11]. In our notation they are given by [46]

first supersymmetry, parameter ϵ_1

$$\begin{aligned}
\delta_1 A &= \sqrt{2}\epsilon_1\psi \\
\delta_1 \psi^\alpha &= \sqrt{2}\epsilon_1^\alpha E \\
\delta_1 E &= 0
\end{aligned} \tag{3.2}$$

[1] We use the same symbol E for the electric field and for the dummy field. Its meaning will be clear from the context.

$$\begin{aligned}
\delta_1 E^\dagger &= i\sqrt{2}\epsilon_1 \not{\partial}\bar{\psi} \\
\delta_1 \bar{\psi}_{\dot{\alpha}} &= -i\sqrt{2}\epsilon_1^\alpha \not{\partial}_{\alpha\dot{\alpha}} A^\dagger \\
\delta_1 A^\dagger &= 0
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
\delta_1 \lambda^\alpha &= -\epsilon_1^\beta (\sigma^{\mu\nu}{}_\beta{}^\alpha v_{\mu\nu} - i\delta_\beta^\alpha D) \\
\delta_1 v^\mu &= i\epsilon_1 \sigma^\mu \bar{\lambda} \quad \delta_1 D = -\epsilon_1 \not{\partial}\bar{\lambda} \\
\delta_1 \bar{\lambda}_{\dot{\alpha}} &= 0
\end{aligned} \tag{3.4}$$

second supersymmetry, parameter ϵ_2

$$\begin{aligned}
\delta_2 A &= -\sqrt{2}\epsilon_2 \lambda \\
\delta_2 \lambda^\alpha &= -\sqrt{2}\epsilon_2^\alpha E^\dagger \\
\delta_2 E^\dagger &= 0
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
\delta_2 E &= -i\sqrt{2}\epsilon_2 \not{\partial}\bar{\lambda} \\
\delta_2 \bar{\lambda}_{\dot{\alpha}} &= i\sqrt{2}\epsilon_2^\alpha \not{\partial}_{\alpha\dot{\alpha}} A^\dagger \\
\delta_2 A^\dagger &= 0
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
\delta_2 \psi^\alpha &= -\epsilon_2^\beta (\sigma^{\mu\nu}{}_\beta{}^\alpha v_{\mu\nu} + i\delta_\beta^\alpha D) \\
\delta_2 v^\mu &= i\epsilon_2 \sigma^\mu \bar{\psi} \quad \delta_2 D = \epsilon_2 \not{\partial}\bar{\psi} \\
\delta_2 \bar{\psi}_{\dot{\alpha}} &= 0
\end{aligned} \tag{3.7}$$

We note here that by R-symmetry we can obtain the second set of transformations by the first one, by simply replacing $1 \to 2$, $\psi \leftrightarrow -\lambda$, $v^\mu \to -v^\mu$ and $E^\dagger \leftrightarrow E$ in the first set.

The N=2 Yang-Mills low-energy effective Lagrangian, up to second derivatives of the fields and four fermions is given by [47]

$$
\begin{aligned}
\mathcal{L} &= \frac{\mathrm{Im}}{4\pi}\Big(-\mathcal{F}''(A)[\partial_\mu A^\dagger \partial^\mu A + \frac{1}{4} v_{\mu\nu}\hat{v}^{\mu\nu} + i\psi \not{\partial}\bar{\psi} + i\lambda \not{\partial}\bar{\lambda} - (EE^\dagger + \frac{1}{2}D^2)] \\
&+ \mathcal{F}'''(A)[\frac{1}{\sqrt{2}}\lambda\sigma^{\mu\nu}\psi v_{\mu\nu} - \frac{1}{2}(E^\dagger\psi^2 + E\lambda^2) + \frac{i}{\sqrt{2}}D\psi\lambda] + \mathcal{F}''''(A)[\frac{1}{4}\psi^2\lambda^2]\Big)
\end{aligned}
\tag{3.8}
$$

where $\mathcal{F}(A)$ is the prepotential discussed in the last Chapter, the prime indicates derivative with respect to the scalar field; the fields appearing are the ones remaining massless after SSB, they describe the whole effective dynamics; $v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$ is the Abelian vector field strength, $v^*_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}v^{\rho\sigma}$ is its dual, $\hat{v}_{\mu\nu} = v_{\mu\nu} + \frac{i}{2}v^*_{\mu\nu}$ is its self-dual projection and $\hat{v}^\dagger_{\mu\nu} = v_{\mu\nu} - \frac{i}{2}v^*_{\mu\nu}$ its anti-self-dual projection. Note that if we define the electric and magnetic fields as usual, $E^i = v^{0i}$ and $B^i = \frac{1}{2}\epsilon^{0ijk}v_{jk}$, respectively, we have $\hat{v}^{0i} = E^i + iB^i$ and $\hat{v}^{\dagger 0i} = E^i - iB^i$. Susy constraints all the fields to be in the same representation of the gauge group as the vector field, namely the adjoint representation. In the U(1) case this representation is trivial, and the derivatives are standard rather than covariant. We notice here that v_0 plays the role of a Lagrange multiplier, and the associate constraint is the Gauss law. Thus, by taking the derivative of $\mathcal{L}$ with respect to v_0 we obtain the quantum modified Gauss law for this theory, namely

$$
0 = \frac{\partial \mathcal{L}}{\partial v_0} = \partial_i \Pi^i \tag{3.9}
$$

where $\Pi^i = \partial\mathcal{L}/\partial(\partial_0 v_i)$ is the conjugate momentum of v_i, given by

$$
\Pi^i = -(\mathcal{I}E^i - \mathcal{R}B^i) + \frac{1}{i\sqrt{2}}(\mathcal{F}'''\lambda\sigma^{0i}\psi - \mathcal{F}^{\dagger\,'''}\bar{\lambda}\bar{\sigma}^{0i}\bar{\psi}) \tag{3.10}
$$

and $\mathcal{F} = \mathcal{R} + i\mathcal{I}$.

3.2 The classical case

We shall study, for the moment, the classical limit of this Lagrangian. At this end it is sufficient to recall that in the classical case there is no running of the coupling constant, therefore there is only one global description at any scale of the energy. Thus we can use the expression (2.30) to write the classical limit as

$$\mathcal{F}(A) \to \frac{1}{2}\tau A^2 \tag{3.11}$$

where τ is the complex coupling constant already introduced

$$\tau = \tau_R + i\tau_I = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2} \tag{3.12}$$

In this limit the second line of (3.8) vanishes and the first line becomes

$$\mathcal{L} = \frac{\mathrm{Im}}{4\pi}\Big(-\tau[\partial_\mu A^\dagger \partial^\mu A + \frac{1}{4}\upsilon_{\mu\nu}\hat{\upsilon}^{\mu\nu} + i\psi\ \partial\!\!\!/ \bar{\psi} + i\lambda\ \partial\!\!\!/ \bar{\lambda} - (EE^\dagger + \frac{1}{2}D^2)]\Big) \tag{3.13}$$

By using $\mathrm{Im}(zw) = z_I w_R + z_R w_I = \frac{1}{2i}(zw - z^* w^*)$ we can write explicitly this Lagrangian as

$$\begin{aligned}\mathcal{L} &= -\frac{1}{g^2}\Big(\frac{1}{4}\upsilon_{\mu\nu}\upsilon^{\mu\nu} + \partial_\mu A^\dagger \partial^\mu A - (EE^\dagger + \frac{1}{2}D^2)\Big) - \frac{\theta}{64\pi^2}\upsilon_{\mu\nu}\upsilon^{*\mu\nu} \\ &\quad -\frac{1}{4\pi}\Big(\frac{\tau}{2}\psi\ \partial\!\!\!/ \bar{\psi} - \frac{\tau^*}{2}\bar{\psi}\bar{\partial\!\!\!/}\psi + \frac{\tau}{2}\lambda\ \partial\!\!\!/ \bar{\lambda} - \frac{\tau^*}{2}\bar{\lambda}\bar{\partial\!\!\!/}\lambda\Big)\end{aligned} \tag{3.14}$$

The non canonical momenta from the Lagrangian (3.14) are given by

$$\pi^\mu_A \equiv \frac{\partial \mathcal{L}}{\partial\, \partial_\mu A} = -\frac{1}{g^2}\partial^\mu A^\dagger \tag{3.15}$$

$$\pi^\mu_{A^\dagger} \equiv \frac{\partial \mathcal{L}}{\partial\, \partial_\mu A^\dagger} = -\frac{1}{g^2}\partial^\mu A = (\pi^\mu_A)^\dagger \tag{3.16}$$

$$\Pi^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial\, \partial_\mu \upsilon_\nu} = -\frac{1}{g^2}\upsilon^{\mu\nu} - \frac{\theta}{16\pi^2}\upsilon^{*\mu\nu} = -\frac{1}{8\pi i}(\tau\hat{\upsilon}^{\mu\nu} - \tau^*\hat{\upsilon}^{\dagger\mu\nu}) \tag{3.17}$$

and

$$4\pi(\pi^\mu_{\bar{\psi}})_{\dot{\alpha}} = \frac{\tau}{2}\psi^\alpha\sigma^\mu_{\alpha\dot{\alpha}} \quad 4\pi(\pi^\mu_\psi)^\alpha = -\frac{\tau^*}{2}\bar{\psi}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha} \tag{3.18}$$

$$4\pi(\pi^\mu_{\bar{\lambda}})_{\dot{\alpha}} = \frac{\tau}{2}\lambda^\alpha\sigma^\mu_{\alpha\dot{\alpha}} \quad 4\pi(\pi^\mu_\lambda)^\alpha = -\frac{\tau^*}{2}\bar{\lambda}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha} \tag{3.19}$$

where $\pi^\mu_\chi \equiv \frac{\partial\mathcal{L}}{\partial\,\partial_\mu\chi}$, $\chi = \psi, \lambda, \bar\psi, \bar\lambda$.

Now we want to compute the Susy ♡ Noether currents for this theory. In fact we have only to compute the first Susy current, since by R-symmetry, charge conjugation and complex conjugation we can obtain the other currents. As explained in the first Chapter, it is matter to compute the V_μ part of the current. In the classical case this is an easy matter. Thus, by taking the variation *off-shell* of (3.14) under the first Susy transformations in (3.2)-(3.4), $\delta_1\mathcal{L} = \partial_\mu V^\mu_1$, we obtain

$$\begin{aligned} V^\mu_1 &= \pi^\mu_A \delta_1 A + \Pi^{\mu\nu}\delta_1 v_\nu \\ &\quad + \frac{\tau^*}{\tau}\delta_1\bar\psi\pi^\mu_{\bar\psi} + \delta_1\psi\pi^\mu_\psi + \delta^D_1\lambda\pi^\mu_\lambda + \frac{\tau}{\tau^*}\delta^v_1\lambda\pi^\mu_\lambda \end{aligned} \tag{3.20}$$

where again $\delta^X Y$ stands for the term in the variation of Y that contains X, for instance $\delta^D_1\lambda^\alpha \equiv i\epsilon^\alpha_1 D$. The total current J^μ_1 is then given by[2]

$$\begin{aligned} J^\mu_1 &= N^\mu_1 - V^\mu_1 \\ &= \pi^\mu_A\delta_1 A + \Pi^{\mu\nu}\delta_1 v_\nu + \delta_1\psi\pi^\mu_\psi + \delta_1\lambda\pi^\mu_\lambda + \delta_1\bar\psi\pi^\mu_{\bar\psi} - V^\mu_1 \\ &= \frac{2i\tau_I}{\tau}\delta_1\bar\psi\pi^\mu_{\bar\psi} - \frac{2i\tau_I}{\tau^*}\delta^{\rm on}_1\lambda\pi^\mu_\lambda \end{aligned} \tag{3.21}$$

where N^μ_1 is the rigid current, $\delta_1\bar\lambda = 0$ and $\delta^{\rm on}_1\lambda$ stand for the variation of λ with dummy fields on-shell (there are no dummy fields in the variation of $\bar\psi$). In this case this means $E = D = 0$ and one could also wonder if they are simply canceled in the total current. But, in agreement with our recipe, we shall see later that indeed the dummy fields, and *only* them, have been automatically projected on-shell.

[2]We choose to explicitly keep the Susy parameters ϵ^α_L, $L = 1, 2$. This simplifies some computations involving spinors. Therefore J^μ_1 stands for $\epsilon^1 J^\mu_1$ and also for $\epsilon_1 J^{1\mu}$. In the following we shall not keep track of the position of these indices, they will be treated as labels.

If we set $\theta = 0$ in this non-canonical setting, we recover the same type of expression, $J^\mu = 2N^\mu_{\text{fermi}}$, for the total current obtained in the massive WZ model, namely

$$J^\mu_1|_{\theta=0} = 2(\delta_1^{\text{on}}\lambda\pi^\mu_\lambda + \delta_1\bar{\psi}\pi^\mu_{\bar{\psi}}) \tag{3.22}$$

Once again we see that the double counting of the fermionic degrees of freedom provides a very compact formula for the currents. All the informations are contained in the fermionic sector, since the variations of the fermions contain the bosonic momenta. Unfortunately this does not seem to be the case for the effective theory, as we shall see in the next Section.

We have now to integrate by parts in the fermionic sector of the Lagrangian (3.14) to obtain a proper phase space. Everything proceeds along the same lines as for the WZ model. The fermionic sector becomes

$$\mathcal{L}^I_{\text{fermi}} = -i\frac{1}{g^2}(\psi\ \partial\!\!\!/\bar{\psi} + \lambda\ \partial\!\!\!/\bar{\lambda}) \tag{3.23}$$

where with I we indicate one of the two possible choices ($\bar{\psi}$ and $\bar{\lambda}$ are the fields). Thus the canonical fermionic momenta are

$$(\pi^{I\mu}_{\bar{\psi}})_{\dot{\alpha}} = \frac{i}{g^2}\psi^\alpha\sigma^\mu_{\alpha\dot{\alpha}} \quad (\pi^{I\mu}_{\bar{\lambda}})_{\dot{\alpha}} = \frac{i}{g^2}\lambda^\alpha\sigma^\mu_{\alpha\dot{\alpha}} \tag{3.24}$$

and $\pi^{I\mu}_\psi = \pi^{I\mu}_\lambda = 0$. In this case there is no effect of the partial integration on the bosonic momenta since $\partial_\mu\tau = 0$, we shall see that this is not longer the case for the effective theory. Also, the partial integration changes V^μ, but, of course, also N^μ changes accordingly and they still combine to give the same total current J^μ. Namely

$$\begin{aligned} N^{I\mu}_1 &= \pi^\mu_A\delta_1 A + \Pi^{\mu\nu}\delta_1 v_\nu + \delta_1\bar{\psi}\pi^\mu_{\bar{\psi}} & (3.25)\\ V^{I\mu}_1 &= \pi^\mu_A\delta_1 A + \frac{1}{8\pi i}\tau^*\epsilon_1\sigma_\nu\bar{\lambda}v^{*\mu\nu} & (3.26)\end{aligned}$$

and

$$\begin{aligned} J_1^{I\mu} &= N_1^{I\mu} - V_1^{I\mu} \\ &= \delta_1 \bar{\psi} \pi_{\bar{\psi}}^{I\mu} - \frac{i}{g^2} \epsilon_1 \sigma_\nu \bar{\lambda} \hat{v}^{\mu\nu} \qquad (3.27) \\ &= -\sqrt{2} \epsilon_1 \sigma^\nu \bar{\sigma}^\mu \pi_{\nu A} + \Pi^{\mu\nu} \delta_1 v_\nu - \frac{1}{8\pi i} \tau^* \epsilon_1 \sigma_\nu \bar{\lambda} v^{*\mu\nu} \qquad (3.28) \end{aligned}$$

where we used the identities

$$\delta_1 \bar{\psi} \pi_{\bar{\psi}}^{I\mu} = -\sqrt{2} \epsilon_1 \sigma^\nu \bar{\sigma}^\mu \pi_{\nu A} \qquad (3.29)$$

$$-\frac{i}{g^2} \epsilon_1 \sigma_\nu \bar{\lambda} \hat{v}^{\mu\nu} = \Pi^{\mu\nu} \delta_1 v_\nu - \frac{1}{8\pi i} \tau^* \epsilon_1 \sigma_\nu \bar{\lambda} v^{*\mu\nu} \qquad (3.30)$$

The point we make here is that the current is once and for all given by

$$J_1^\mu = \frac{1}{g^2} (\sqrt{2} \epsilon_1 \sigma^\nu \bar{\sigma}^\mu \partial_\nu A^\dagger - i \epsilon_1 \sigma_\nu \bar{\lambda} \hat{v}^{\mu\nu}) \qquad (3.31)$$

but its content in terms of canonical variables changes according to partial integration. Furthermore one has to conveniently re-express the current obtained via Noether procedure to obtain the expression (3.27) or (3.28) in terms of bosonic or fermionic momenta and transformations, respectively. Note also that θ does not appear in the explicit formula, as could be expected.

The next step is to choose a gauge for the vector field and define the conjugate momenta (remember that the metric is given by $\eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$). We shall work in the temporal gauge for the vector field $v^0 = 0$, the conjugate momenta are then given by

$$\pi_A \equiv \pi_A^0 = -\frac{1}{g^2} \partial^0 A^\dagger \quad \pi_{A^\dagger} \equiv \pi_{A^\dagger}^0 = -\frac{1}{g^2} \partial^0 A \qquad (3.32)$$

$$\Pi^i \equiv \Pi^{0i} = -\frac{1}{8\pi i} (\tau \hat{v}^{0i} - \tau^* \hat{v}^{\dagger 0i}) \qquad (3.33)$$

and[3]

$$\pi^I_{\bar\psi} \equiv \frac{i}{g^2}\psi\sigma^0 \quad \pi^I_{\bar\lambda} \equiv \frac{i}{g^2}\lambda\sigma^0 \tag{3.34}$$

With this choice the first Susy charge is given by

$$\epsilon_1 Q_1^I \equiv \int d^3x J_1^{I0} = \frac{1}{g^2}\int d^3x(\sqrt{2}\epsilon_1\sigma^\nu\bar\sigma^0\psi\partial_\nu A^\dagger - i\epsilon_1\sigma_i\bar\lambda\hat{v}^{0i}) \tag{3.35}$$

$$= \int d^3x(\delta_1\bar\psi\pi^I_{\bar\psi} - \frac{i}{g^2}\epsilon_1\sigma_i\bar\lambda\hat{v}^{0i}) \tag{3.36}$$

$$= \int d^3x(\delta_1 A\pi_A + \sqrt{2}\epsilon_1\sigma^i\bar\sigma^0\psi\partial_i A^\dagger + \delta_1 v_i\Pi^i + \frac{i\tau^*}{8\pi}\epsilon_1\sigma_i\bar\lambda v^{*0i}) \tag{3.37}$$

The other charges are obtained by R-symmetry, charge conjugation and complex conjugation. They are given by

$$\epsilon_2 Q_2^I \equiv \int d^3x J_2^{I0} = \frac{1}{g^2}\int d^3x(-\sqrt{2}\epsilon_2\sigma^\nu\bar\sigma^0\lambda\partial_\nu A^\dagger - i\epsilon_2\sigma_i\bar\psi\hat{v}^{0i}) \tag{3.38}$$

$$\bar\epsilon_1\bar Q_1^I \equiv \int d^3x J_1^{I\dagger 0} = \frac{1}{g^2}\int d^3x(\sqrt{2}\bar\epsilon_1\bar\sigma^\nu\sigma^0\bar\psi\partial_\nu A - i\bar\epsilon_1\bar\sigma_i\lambda\hat{v}^{\dagger 0i}) \tag{3.39}$$

$$\bar\epsilon_2\bar Q_2^I \equiv \int d^3x J_2^{I\dagger 0} = \frac{1}{g^2}\int d^3x(-\sqrt{2}\bar\epsilon_2\bar\sigma^\nu\sigma^0\bar\lambda\partial_\nu A - i\bar\epsilon_2\bar\sigma_i\psi\hat{v}^{\dagger 0i}) \tag{3.40}$$

Of course one needs to rearrange also these expressions in terms of conjugate momenta and fields transformations as we did in (3.36) and (3.37) for the first Susy charge.

3.2.1 Transformations and Hamiltonian from the Q_α's

We now want to test the correctness of these charges by commuting them to obtain the Susy transformations of the fields and the Hamiltonian. At this end we first introduce the basic non zero equal-time graded Poisson brackets, given by (see also Appendix C)

$$\{A(x), \pi_A(y)\}_- = \{A^\dagger(x), \pi^\dagger_A(y)\}_- = \delta^{(3)}(\vec{x}-\vec{y}) \tag{3.41}$$

$$\{v_i(x), \Pi^j(y)\}_- = \delta^j_i\delta^{(3)}(\vec{x}-\vec{y}) \tag{3.42}$$

[3]See Appendix C on the conventions for these momenta.

and

$$\{\bar{\psi}_{\dot{\alpha}}(x), \pi^{\dot{\beta}}_{\bar{\psi}}(y)\}_+ = \{\bar{\lambda}_{\dot{\alpha}}(x), \pi^{\dot{\beta}}_{\bar{\lambda}}(y)\}_+ = \delta^{\dot{\beta}}_{\dot{\alpha}} \delta^{(3)}(\vec{x} - \vec{y}) \tag{3.43}$$

Due to the conventions used for the graded Poisson brackets, for the fermions we act with the charge from the left while for the bosons we act from the right. We shall call Δ_1 the transformation induced by our charge $\epsilon_1 Q_1^I$ in (3.35). For the bosonic transformations we use the expression (3.37), whereas for the fermions we use the expression (3.36). Thus we obtain

$$\begin{aligned}
\Delta_1 A(x) &\equiv \{A(x)\ ,\ \epsilon_1 Q_1^I\}_- = \int d^3y \{A(x)\ ,\ \sqrt{2}\epsilon_1 \psi(y) \pi_A(y) + \text{irr.}\}_- \\
&= \sqrt{2}\epsilon_1 \psi(x) = \delta_1 A(x) &(3.44)\\
\Delta_1 A^\dagger(x) &\equiv \{A^\dagger(x)\ ,\ \epsilon_1 Q_1^I\}_- = 0 = \delta_1 A^\dagger(x) &(3.45)\\
\Delta_1 v_i(x) &\equiv \{v_i(x)\ ,\ \epsilon_1 Q_1^I\}_- \\
&= \int d^3y \{v_i(x)\ ,\ \Pi^j(y)\}_- \delta_1 v_j(y) = \delta_1 v_i(x) &(3.46)\\
\Delta_1 \bar{\psi}_{\dot{\alpha}}(x) &\equiv \{\epsilon_1 Q_1^I\ ,\ \bar{\psi}_{\dot{\alpha}}(x)\}_- \\
&= \int d^3y \delta_1 \bar{\psi}_{\dot{\beta}}(y) \{\pi^{I\dot{\beta}}_{\bar{\psi}}(y)\ ,\ \bar{\psi}_{\dot{\alpha}}(x)\}_+ = \delta_1 \bar{\psi}_{\dot{\alpha}}(x) &(3.47)\\
\Delta_1 \bar{\lambda}_{\dot{\alpha}}(x) &\equiv \{\epsilon_1 Q_1^I\ ,\ \bar{\lambda}_{\dot{\alpha}}(x)\}_- = 0 = \delta_1 \bar{\lambda}_{\dot{\alpha}}(x) &(3.48)
\end{aligned}$$

where "irr." stands for terms irrelevant for the Poisson brackets. The transformations for ψ and λ have to be obtained by acting with the charge on the conjugate momenta of $\bar{\psi}$ and $\bar{\lambda}$, respectively

$$\Delta_1 \pi^I_{\bar{\psi}\dot{\alpha}}(x) \equiv \{\epsilon_1 Q_1^I\ ,\ \pi^I_{\bar{\psi}\dot{\alpha}}(x)\}_- = 0 \tag{3.49}$$

since $\pi^I_{\bar{\psi}\dot{\alpha}} = \frac{i}{g^2}\sigma^0_{\alpha\dot{\alpha}}\psi^\alpha$ we have $\Delta_1 \psi^\alpha = 0 = \delta_1^{\text{on}} \psi^\alpha$. For λ we have

$$\begin{aligned}
\Delta_1 \pi^I_{\bar{\lambda}\dot{\alpha}}(x) &\equiv \{\epsilon_1 Q_1^I\ ,\ \pi^I_{\bar{\lambda}\dot{\alpha}}(x)\}_- = \int d^3y (-\frac{i}{g^2}\epsilon_1^\alpha \sigma_{i\alpha\dot{\beta}} \{\bar{\lambda}^{\dot{\beta}}(y)\ ,\ \pi_{\bar{\lambda}\dot{\alpha}}(x)\}_+ \\
&= -\frac{i}{g^2}\epsilon_1^\alpha \sigma_{i\alpha\dot{\alpha}} \hat{v}^{0i}(x) &(3.50)
\end{aligned}$$

by multiplying both sides by $\bar{\sigma}^{0\dot{\alpha}\beta}$ and using the identities given in Appendix B we obtain

$$\Delta_1 \lambda^\beta(x) = -(\epsilon_1 \sigma^{\mu\nu})^\beta v_{\mu\nu} = \delta^{\text{on}} \lambda^\beta(x) \tag{3.51}$$

Note that, due to the gauge chosen for the vector field, we only reproduce the transformations up to v^0.

Thus $\Delta_1 \equiv \delta_1^{\text{on}}$. By a similar computation, that we shall not write down here, we see that the same happens for $\bar{\Delta}_1$, Δ_2 and $\bar{\Delta}_2$. Therefore our charges are the correct ones, in the spirit described in Chapter 1. Note that we did not need to improve the current (therefore the charge) in order to produce the right transformations.

One could also check that the Hamiltonian obtained by Legendre transforming the Lagrangian agrees with the one obtained from our charges.

The Susy algebra for N=2, introduced in Chapter 2, in terms of Poisson brackets is given by[4]

$$\begin{aligned}
\{Q_{L\alpha} \ , \ \bar{Q}_{M\dot{\alpha}}\}_+ &= 2i \ \sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta_{LM} && (3.52)\\
\{Q_{L\alpha} \ , \ Q_{M\beta}\}_+ &= 2i \ Z \epsilon_{\alpha\beta} \epsilon_{LM} && (3.53)\\
\{\bar{Q}_{L\dot{\alpha}} \ , \ \bar{Q}_{M\dot{\beta}}\}_+ &= 2i \ Z^* \epsilon_{\dot{\alpha}\dot{\beta}} \epsilon_{LM} && (3.54)
\end{aligned}$$

where $L, M = 1, 2$. The Hamiltonian is then simply obtained as

$$H = -\frac{i}{4} \bar{\sigma}^{0\dot{\alpha}\alpha} \{Q_{L\alpha} \ , \ \bar{Q}_{L\dot{\alpha}}\}_+ \tag{3.55}$$

where $L = 1$ or $L = 2$, and we define $H = P^0 = -P_0$.

We shall not write down the details of this easy computation here, since in the next Chapter we shall spend some time on the effective Hamiltonian of the SU(2) effective theory. The result of the classical computation,

[4]Note that we keep all the indices L, M in the lower position. This reflects our choice to work with $\epsilon_L Q_L$ as explained earlier.

performed for $\theta = 0$, is

$$H = \int d^3x[g^2(\pi_A\pi_A^\dagger + \frac{1}{2}\vec{E}^2) + \frac{1}{g^2}(\vec{\nabla}A\cdot\vec{\nabla}A^\dagger + \frac{1}{2}\vec{B}^2 - i\psi\not{\nabla}\bar{\psi} - i\lambda\not{\nabla}\bar{\lambda})] \quad (3.56)$$

where $E^i = v^{0i}$ and $B^i = \frac{1}{2}\epsilon^{0ijk}v_{jk}$. The same result is obtained by Legendre transforming the Lagrangian (3.14) for $\theta = 0$.

3.2.2 The central charge

We can now compute the central charge Z for the classical theory from the algebra above given. Let us start by computing the six terms (three pairs) contributing to the centre

$$\begin{aligned}
\{\epsilon_1 Q_1, \epsilon_2 Q_2\}_- &= \int d^3x d^3y \Big(\frac{i}{8\pi}\{\Pi^i, v^{*0j}\}_- \delta_1 v_i \tau^* \epsilon_2 \sigma_j \bar{\psi} \\
&+ \frac{i}{8\pi}\{v^{*0i}, \Pi^j\}_- \delta_2 v_j \tau^* \epsilon_1 \sigma_i \bar{\lambda} && (3.57) \\
&+ \Pi^i \delta_2 \bar{\lambda}_{\dot{\alpha}} \{\delta_1 v_i, \pi_{\bar{\lambda}}^{\dot{\alpha}}\}_- \\
&+ \Pi^j \delta_1 \bar{\psi}_{\dot{\alpha}} \{\pi_{\bar{\psi}}^{\dot{\alpha}}, \delta_2 v_j\}_- && (3.58) \\
&+ \frac{i}{8\pi}\delta_1 \bar{\psi}_{\dot{\alpha}} \{\pi_{\bar{\psi}}^{\dot{\alpha}}, \epsilon_2 \sigma_j \bar{\psi}\}_- \tau^* v^{*0j} \\
&+ \frac{i}{8\pi}\delta_2 \bar{\lambda}_{\dot{\alpha}} \{\epsilon_1 \sigma_i \bar{\lambda}, \pi_{\bar{\lambda}}^{\dot{\alpha}}\}_- \tau^* v^{*0i} \Big) && (3.59)
\end{aligned}$$

On the one hand, terms (3.57) give

$$\frac{i}{4\pi}\tau^* \int d^3x \partial_i (\epsilon_1 \sigma^0 \bar{\psi} \epsilon_2 \sigma^i \bar{\lambda} - \epsilon_2 \sigma^0 \bar{\lambda} \epsilon_1 \sigma^i \bar{\psi}) \quad (3.60)$$

On the other hand, terms (3.58) and (3.59) give

$$\sqrt{2}\epsilon_1\epsilon_2 \int d^3x \Big(\partial_i (2\Pi^i A^\dagger + \frac{1}{4\pi} v^{*0i} A_D^\dagger) + 2(\partial_i \Pi^i) A^\dagger + \frac{1}{4\pi}(\partial_i v^{*0i}) A_D^\dagger \Big) \quad (3.61)$$

where $A_D^\dagger = \tau^* A^\dagger$ is the classical analogue of the dual of the scalar field, as discussed in the previous Chapter.

Getting rid of ϵ_1 and ϵ_2 and summing over the spinor indices ($\{\epsilon_1 Q_1, \epsilon_2 Q_2\}_- = -\epsilon_1^\alpha \epsilon_2^\beta \{Q_{1\alpha}, Q_{2\beta}\}_+$ and $\epsilon_{\alpha\beta}\epsilon^{\alpha\beta} = -2$) we can write the centre as

$$Z = \frac{i}{4}\epsilon^{\alpha\beta}\{Q_{1\alpha}, Q_{2\beta}\}_+ \tag{3.62}$$

obtaining

$$\begin{aligned} Z &= \int d^3x \Big(\partial_i [i\sqrt{2}(\Pi^i A^\dagger + \frac{1}{4\pi} B^i A_D^\dagger) - \frac{1}{4\pi}\tau^* \bar{\psi}\bar{\sigma}^{i0}\bar{\lambda}] \\ &+ i\sqrt{2}[(\partial_i \Pi^i) A^\dagger + \frac{1}{4\pi}(\partial_i v^{*0i}) A_D^\dagger] \Big) \end{aligned} \tag{3.63}$$

By using the Bianchi identities, $\partial_i v^{*0i} = 0$, and the classical limit of the Gauss law (3.9), we are left with a total divergence. The final expression for Z is then given by

$$Z = i\sqrt{2} \int d^2\vec{\Sigma} \cdot (\vec{\Pi} A^\dagger + \frac{1}{4\pi}\vec{B} A_D^\dagger) \tag{3.64}$$

where $d^2\vec{\Sigma}$ is the measure on the sphere at infinity S^2_∞, and we have made the usual assumption that $\bar{\psi}$ and $\bar{\lambda}$ fall off at least like $r^{-\frac{3}{2}}$. This is the classical result discussed in the previous Chapter. Note that we ended up with the anti-holomorphic centre.

When we define the electric and magnetic charges *à la* Witten and Olive, we can write

$$Z = i\sqrt{2}(n_e a^* + n_m a_D^*) \tag{3.65}$$

where $< 0|A^\dagger|0 >= a^*$ and n_e, n_m are the electric and magnetic quantum numbers, respectively.

3.3 The effective case

We now want to move to the interesting case of the effective theory described by the Lagrangian (3.8). Let us write down this Lagrangian explicitly

$$\mathcal{L} = \frac{1}{2i}\Big[-\mathcal{F}''(A)[\partial_\mu A^\dagger \partial^\mu A + \frac{1}{4}v_{\mu\nu}\hat{v}^{\mu\nu} + i\psi \not{\partial}\bar{\psi} + i\lambda \not{\partial}\bar{\lambda} - (EE^\dagger + \frac{1}{2}D^2)]$$

$$
\begin{aligned}
&+\mathcal{F}'''(A)[\frac{1}{\sqrt{2}}\lambda\sigma^{\mu\nu}\psi v_{\mu\nu} - \frac{1}{2}(E^\dagger\psi^2 + E\lambda^2) + \frac{i}{\sqrt{2}}D\psi\lambda] \\
&+\mathcal{F}''''(A)[\frac{1}{4}\psi^2\lambda^2] \\
+\quad & \mathcal{F}^{\dagger\prime\prime}(A^\dagger)[\partial_\mu A^\dagger\partial^\mu A + \frac{1}{4}v_{\mu\nu}\hat{v}^{\dagger\mu\nu} + i\bar{\psi}\bar{\partial\!\!\!/}\psi + i\bar{\lambda}\bar{\partial\!\!\!/}\lambda - (EE^\dagger + \frac{1}{2}D^2)] \\
&+\mathcal{F}^{\dagger\prime\prime\prime}(A^\dagger)[\frac{1}{\sqrt{2}}\bar{\psi}\bar{\sigma}^{\mu\nu}\bar{\lambda}v_{\mu\nu} + \frac{1}{2}(E\bar{\psi}^2 + E^\dagger\bar{\lambda}^2) + \frac{i}{\sqrt{2}}D\bar{\psi}\bar{\lambda}] \\
&-\mathcal{F}^{\dagger\prime\prime\prime\prime}(A^\dagger)[\frac{1}{4}\bar{\psi}^2\bar{\lambda}^2]\Big] \qquad (3.66)
\end{aligned}
$$

where, for the moment, we scale $\mathcal{F}$ by a factor of 4π.

The non canonical momenta are given by

$$
\pi^\mu_A = -\mathcal{I}\partial^\mu A^\dagger \qquad \pi^\mu_{A^\dagger} = (\pi^\mu_A)^\dagger \qquad (3.67)
$$

$$
\Pi^{\mu\nu} = -\frac{1}{2i}(\mathcal{F}''\hat{v}^{\mu\nu} - \mathcal{F}^{\dagger\prime\prime}\hat{v}^{\dagger\mu\nu}) + \frac{1}{i\sqrt{2}}(\mathcal{F}'''\lambda\sigma^{\mu\nu}\psi - \mathcal{F}^{\dagger\prime\prime\prime}\bar{\lambda}\bar{\sigma}^{\mu\nu}\bar{\psi}) \qquad (3.68)
$$

and

$$
\begin{aligned}
(\pi^\mu_{\bar{\psi}})_{\dot{\alpha}} = \frac{1}{2}\mathcal{F}''\psi^\alpha\sigma^\mu_{\alpha\dot{\alpha}} \quad (\pi^\mu_\psi)^\alpha = -\frac{1}{2}\mathcal{F}^{\dagger\prime\prime}\bar{\psi}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha} \qquad (3.69) \\
(\pi^\mu_{\bar{\lambda}})_{\dot{\alpha}} = \frac{1}{2}\mathcal{F}''\lambda^\alpha\sigma^\mu_{\alpha\dot{\alpha}} \quad (\pi^\mu_\lambda)^\alpha = -\frac{1}{2}\mathcal{F}^{\dagger\prime\prime}\bar{\lambda}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha} \qquad (3.70)
\end{aligned}
$$

where $\mathcal{F}'' = \mathcal{R} + i\mathcal{I}$.

This time the dummy fields couple non trivially to the fermions. Their expression on-shell is given by

$$
\begin{aligned}
D &= -\frac{1}{2\sqrt{2}}(f\psi\lambda + f^\dagger\bar{\psi}\bar{\lambda}) \qquad (3.71) \\
E^\dagger &= -\frac{i}{4}(f\lambda^2 - f^\dagger\bar{\psi}^2) \qquad (3.72) \\
E &= \frac{i}{4}(f^\dagger\bar{\lambda}^2 - f\psi^2) \qquad (3.73)
\end{aligned}
$$

where $f(A, A^\dagger) \equiv \mathcal{F}'''/\mathcal{I}$.

As in the classical case we can concentrate on the computation of the first Susy current J^μ_1.The task, of course, is to find V^μ_1. It turns out that its computation is by no means easy as shown in some details in Appendix D

3.3.1 Computation of the effective J_1^μ

To compute V_1^μ we first realize that, by varying $\mathcal{L}$ off-shell under δ_1 given in (3.2)-(3.4), there is no mixing of the $\mathcal{F}(A)$ terms with the $\mathcal{F}^\dagger(A^\dagger)$ terms. The structure of the Lagrangian (3.66) is

$$\mathcal{L} \equiv \frac{1}{2i}\Big[\{-\mathcal{F}'' \cdot [1] + \mathcal{F}''' \cdot [2] + \mathcal{F}'''' \cdot [3]\} - \{h.c.\}\Big] \tag{3.74}$$

The terms [1] are bilinear in the fermions and in the bosons ($[1] \sim [2F+2B]$), the terms [2] are products of terms bilinear in the fermions and linear in the bosons ($[2] \sim [2F \cdot 1B]$), finally the terms [3] are quadrilinear in the fermions ($[3] \sim [4F]$). When we vary the $\mathcal{F}$ terms under δ_1 we see that $(\delta_1 \mathcal{F}'''')[3] = 0$, whereas

$$\begin{array}{rcl} \mathcal{F}''''\delta_1[3] \sim \mathcal{F}''''(1B \cdot 3F) & \text{combines with} & (\delta_1\mathcal{F}''')[2] \sim \mathcal{F}''''(1B \cdot 3F) \\ \mathcal{F}'''\delta_1[2] \sim \mathcal{F}'''(2B \cdot 1F + 3F) & \text{combines with} & (\delta_1\mathcal{F}'')[1] \sim \mathcal{F}'''(2B \cdot 1F + 3F) \end{array}$$

Similarly for the $\mathcal{F}^\dagger$ terms. The aim is to write these variations as one single total divergence and express it in terms of momenta and variations of the fields.

The ϵ_1 computation, illustrated in Appendix D, gives

$$\begin{aligned} V_1^\mu \quad = \quad & \delta_1 A \pi_A^\mu + \frac{\mathcal{F}^{\dagger\prime\prime}}{\mathcal{F}''}\delta_1\bar\psi\pi_{\bar\psi}^\mu + \delta_1\psi\pi_\psi^\mu + \delta_1\lambda\pi_\lambda^\mu \\ & + \frac{1}{2i}\mathcal{F}^{\dagger\prime\prime}\epsilon_1\sigma_\nu\bar\lambda v^{*\mu\nu} + \frac{1}{2\sqrt{2}}\mathcal{F}^{\dagger\prime\prime\prime}\epsilon_1\sigma^\mu\bar\psi\bar\lambda^2 \end{aligned} \tag{3.75}$$

This expression of V_1^μ is far from being a straightforward generalization of the classical one given in (3.20). One could naively try to guess the effective V_1^μ by simply "inverting the arrow" of the classical limit (3.11), $\mathcal{F}'' \leftarrow \tau$, but this is not the case. In fact, there are many other substantial differences. The main three are: the dummy fields now have the quite complicated expressions in terms of functions of the scalar fields (the factors f) and

fermionic bilinears given in (3.71)-(3.73); $\Pi^{\mu\nu}$ does not appear in V_1^μ and it will not be canceled in the total current, as in the classical case (it is now given by the expression (3.68), again with fermionic bilinears and functions of the scalar fields); the last term is an additional quantum factor which one could not have guessed. Of course the rigid current is formally identical to the classical one, namely

$$N_1^\mu = \delta_1 A \pi_A^\mu + \delta_1 v_\nu \Pi^{\mu\nu} + \delta_1 \bar{\psi} \pi_{\bar{\psi}}^\mu + \delta_1 \psi \pi_\psi^\mu + \delta_1 \lambda \pi_\lambda^\mu \tag{3.76}$$

Thus we can write down our total current as

$$\begin{aligned} J_1^\mu &\equiv N_1^\mu - V_1^\mu \\ &= \frac{2i\mathcal{I}}{\mathcal{F}''} \delta_1 \bar{\psi} \pi_{\bar{\psi}}^\mu + \Pi^{\mu\nu} \delta_1 v_\nu - \frac{1}{2i} \mathcal{F}^{\dagger\prime\prime} \epsilon_1 \sigma_\nu \bar{\lambda} v^{*\mu\nu} - \frac{1}{2\sqrt{2}} \mathcal{F}^{\dagger\prime\prime\prime} \epsilon_1 \sigma^\mu \bar{\psi} \bar{\lambda}^2 \end{aligned} \tag{3.77}$$

$$\begin{aligned} &= \sqrt{2}\mathcal{I} \epsilon_1 (\not{\partial} A^\dagger) \bar{\sigma}^\mu \psi - \frac{1}{2i} \mathcal{F}^{\dagger\prime\prime} \epsilon_1 \sigma_\nu \bar{\lambda} v^{*\mu\nu} - \frac{1}{2\sqrt{2}} \mathcal{F}^{\dagger\prime\prime\prime} \epsilon_1 \sigma^\mu \bar{\psi} \bar{\lambda}^2 \\ &\quad + [-\frac{1}{2} (\mathcal{F}'' \hat{v}^{\mu\nu} - \mathcal{F}^{\dagger\prime\prime} \hat{v}^{\dagger\mu\nu}) + \frac{1}{\sqrt{2}} (\mathcal{F}''' \lambda \sigma^{\mu\nu} \psi - \mathcal{F}^{\dagger\prime\prime\prime} \bar{\lambda} \bar{\sigma}^{\mu\nu} \bar{\psi})] \epsilon_1 \sigma_\nu \bar{\lambda} \end{aligned} \tag{3.78}$$

The current (3.77) is not canonically expressed, due to partial integration necessary in the fermionic sector of the kinetic terms in (3.66). Nevertheless if we explicitly write the current in terms of fields and their derivatives as in (3.78) this form will be insensitive to partial integration as we shall show in the next Subsection. We have seen that this is true for simpler cases (the classical theory and the massive WZ toy model of the first Chapter). In the effective case the matter is not trivial.

A final remark is in order. If we conveniently rearrange the terms in (3.78) we can write

$$J_1^\mu = \frac{2i\mathcal{I}}{\mathcal{F}''} \delta_1 \bar{\psi} \pi_{\bar{\psi}}^\mu - \frac{2i\mathcal{I}}{\mathcal{F}''^\dagger} \delta_1^{\text{on}} \lambda \pi_\lambda^\mu + \frac{1}{2\sqrt{2}} \mathcal{F}^{\dagger\prime\prime\prime} \epsilon_1 \sigma^\mu \bar{\psi} \bar{\lambda}^2 + \frac{1}{\sqrt{2}} \mathcal{F}''' \epsilon_1 \psi \lambda \sigma^\mu \bar{\lambda} \tag{3.79}$$

and we see that the "labour saving" formula $J_\mu = 2N^\mu_{\rm fermi}$, no longer holds.

3.3.2 Canonicity

In this Subsection we digress for a moment to establish the canonicity in the effective context. This is a delicate point since it affects not only the definition of the canonical momenta for the fermionic fields, as in the classical case, but even the definition of the canonical momenta for the scalar fields. Let us extract the fermionic kinetic piece from (3.66)

$$\mathcal{L}_{\rm kin.fermi} = \frac{1}{2i}[-\mathcal{F}'' i\psi \not\partial \bar\psi + \mathcal{F}^{\dagger''} i\bar\psi \bar{\not\partial}\psi + (\psi \to \lambda)] \tag{3.80}$$

If we call $\mathcal{L}^I$ the Lagrangian with $\bar\psi$ and $\bar\lambda$ as fields, it differs from $\mathcal{L}$ only in the fermionic kinetic piece and $\mathcal{F}^{\dagger'''}$ type of terms

$$\mathcal{L}^I_{\rm kin.fermi} = \frac{1}{2i}[-\mathcal{F}'' i\psi \not\partial \bar\psi + \mathcal{F}^{\dagger''} i\psi \not\partial\bar\psi - i\mathcal{F}^{\dagger'''}(\partial_\mu A^\dagger)\bar\psi\bar\sigma^\mu\psi + (\psi \to \lambda)] \tag{3.81}$$

Similarly for $\mathcal{L}^{II}$ (ψ and λ as fields)

$$\mathcal{L}^{II}_{\rm kin.fermi} = \frac{1}{2i}[-\mathcal{F}'' i\bar\psi \bar{\not\partial}\psi + \mathcal{F}^{\dagger''} i\bar\psi \not\partial\psi + i\mathcal{F}'''(\partial_\mu A)\psi\sigma^\mu\bar\psi + (\psi \to \lambda)] \tag{3.82}$$

The relation among the three Lagrangians is clearly

$$\begin{aligned} \mathcal{L} &= \mathcal{L}^I + \partial_\mu(\frac{1}{2}\mathcal{F}^{\dagger''}(\bar\psi\bar\sigma^\mu\psi + \bar\lambda\bar\sigma^\mu\lambda)) && (3.83)\\ &= \mathcal{L}^{II} + \partial_\mu(-\frac{1}{2}\mathcal{F}''(\psi\sigma^\mu\bar\psi + \lambda\sigma^\mu\bar\lambda)) && (3.84)\end{aligned}$$

and the momenta change accordingly.

From $\mathcal{L}^I$:

$$\pi^{I\mu}_A = \pi^\mu_A \qquad \pi^{I\mu}_{A^\dagger} = -\mathcal{I}\partial^\mu A - \frac{1}{2}\mathcal{F}^{\dagger'''}(\bar\psi\bar\sigma^\mu\psi + \bar\lambda\bar\sigma^\mu\lambda) \tag{3.85}$$

and

$$\begin{aligned} &(\pi^{I\mu}_{\bar\psi})_{\dot\alpha} = i\mathcal{I}\psi^\alpha\sigma^\mu_{\alpha\dot\alpha} \quad (\pi^{I\mu}_{\psi})^\alpha = 0 \\ &(\pi^{I\mu}_{\bar\lambda})_{\dot\alpha} = i\mathcal{I}\lambda^\alpha\sigma^\mu_{\alpha\dot\alpha} \quad (\pi^{I\mu}_{\lambda})^\alpha = 0 \end{aligned} \tag{3.86}$$

From $\mathcal{L}^{II}$:

$$\pi_A^{II\mu} = -\mathcal{I}\partial^\mu A^\dagger + \frac{1}{2}\mathcal{F}'''(\psi\sigma^\mu\bar{\psi} + \lambda\sigma^\mu\bar{\lambda}) \qquad \pi_{A^\dagger}^{II\mu} = \pi_{A^\dagger}^\mu \tag{3.87}$$

and

$$\begin{aligned}(\pi_{\bar{\psi}}^{II\mu})_{\dot{\alpha}} = 0 \quad (\pi_\psi^{II\mu})^\alpha = i\mathcal{I}\bar{\psi}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha} \\ (\pi_{\bar{\lambda}}^{II\mu})_{\dot{\alpha}} = 0 \quad (\pi_\lambda^{II\mu})^\alpha = i\mathcal{I}\bar{\lambda}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\end{aligned} \tag{3.88}$$

Note that nothing changes for $\Pi^{\mu\nu}$, since

$$\Pi^{I\mu\nu} = \Pi^{II\mu\nu} = \Pi^{\mu\nu} \tag{3.89}$$

whereas in both cases $(\pi_A)^\dagger \neq \pi_{A^\dagger}$. From (3.83) follows that

$$V_1^{\mu I} = V_1^\mu - \frac{1}{2}\mathcal{F}^{\dagger\prime\prime}\delta_1(\bar{\psi}\bar{\sigma}^\mu\psi + \bar{\lambda}\bar{\sigma}^\mu\lambda) \tag{3.90}$$

where $\delta_1\mathcal{L}^I = \partial_\mu V_1^{\mu I}$ and $\delta_1\mathcal{F}^\dagger = 0$. Explicitly (3.90) reads

$$\begin{aligned}V_1^{\mu I} &= \delta_1 A\pi_A^\mu + \frac{\mathcal{F}''^\dagger}{\mathcal{F}''}\delta_1\bar{\psi}\pi_{\bar{\psi}}^\mu + \delta_1\psi\pi_\psi^\mu + \delta_1\lambda\pi_\lambda^\mu \\ &\quad + \frac{1}{2i}\mathcal{F}''^\dagger\epsilon_1\sigma_\nu\bar{\lambda}v^{*\mu\nu} + \frac{1}{2\sqrt{2}}\mathcal{F}'''^\dagger\epsilon_1\sigma^\mu\bar{\psi}\bar{\lambda}^2 \\ &- \frac{1}{2}\mathcal{F}^{\dagger\prime\prime}\delta_1\bar{\psi}\bar{\sigma}^\mu\psi - \frac{1}{2}\mathcal{F}^{\dagger\prime\prime}\bar{\psi}\bar{\sigma}^\mu\delta_1\psi - \frac{1}{2}\mathcal{F}^{\dagger\prime\prime}\bar{\lambda}\bar{\sigma}^\mu\delta_1\lambda\end{aligned} \tag{3.91}$$

the second, third and fourth terms in the first line cancel against the first, second and third terms in the third line respectively, therefore the fermionic momenta are absent from $V_1^{\mu I}$. But also the rigid current changes to

$$N_1^{\mu I} = \delta_1 A\pi_A^\mu + \Pi^{\mu\nu}\delta_1 v_\nu + \delta_1\bar{\psi}\pi_{\bar{\psi}}^{I\mu} \tag{3.92}$$

thus, recalling that $J_1^{\mu I} = N_1^{\mu I} - V_1^{\mu I}$, we have

$$J_1^{\mu I} = \delta_1\bar{\psi}\pi_{\bar{\psi}}^{I\mu} + \Pi^{\mu\nu}\delta_1 v_\nu - \frac{1}{2i}\mathcal{F}''^\dagger\epsilon_1\sigma_\nu\bar{\lambda}v^{*\mu\nu} - \frac{1}{2\sqrt{2}}\mathcal{F}'''^\dagger\epsilon_1\sigma^\mu\bar{\psi}\bar{\lambda}^2 \tag{3.93}$$

which is identical to the one in (3.78) when we write explicitly the transformations and the *new* momenta.

At this point we have to check that the same thing happens with the other partial integration. Here things are slightly more complicated due to the fact that ${\pi_A^{II}}^\mu$ is no longer equal to π_A^μ and $\delta_1\mathcal{F}'' \neq 0$. As we shall see these two problems cancel each other.

First let us look at $V_1^{II\mu}$. From (3.84) follows that

$$
\begin{aligned}
V_1^{II\mu} \quad &= \quad V_1^\mu + \frac{1}{2}\mathcal{F}'''\delta_1 A(\psi\sigma^\mu\bar{\psi} + \lambda\sigma^\mu\bar{\lambda}) + \frac{1}{2}\mathcal{F}''\delta_1(\psi\sigma^\mu\bar{\psi} + \lambda\sigma^\mu\bar{\lambda}) \\
&= \quad \delta_1 A\pi_A^\mu + \frac{\mathcal{F}''^\dagger}{\mathcal{F}''}\delta_1\bar{\psi}\pi_{\bar{\psi}}^\mu + \delta_1\psi\pi_\psi^\mu + \delta_1\lambda\pi_\lambda^\mu \qquad (3.94)\\
&\quad\quad + \frac{1}{2i}\mathcal{F}''^\dagger\epsilon_1\sigma_\nu\bar{\lambda}v^{*\mu\nu} + \frac{1}{2\sqrt{2}}\mathcal{F}'''^\dagger\epsilon_1\sigma^\mu\bar{\psi}\bar{\lambda}^2 \\
&- \quad \frac{1}{2}\mathcal{F}''\delta_1\bar{\psi}\bar{\sigma}^\mu\psi - \frac{1}{2}\mathcal{F}''\bar{\psi}\bar{\sigma}^\mu\delta_1\psi - \frac{1}{2}\mathcal{F}''\bar{\lambda}\bar{\sigma}^\mu\delta_1\lambda \qquad (3.95)\\
&\quad\quad + \delta_1 A\frac{1}{2}\mathcal{F}'''(\psi\sigma^\mu\bar{\psi} + \lambda\sigma^\mu\bar{\lambda}) \qquad (3.96)
\end{aligned}
$$

The first term in (3.94) combines with the last term in (3.96) to give $\delta_1 A{\pi_A^{II}}^\mu$; the second term in (3.94) and the the first in (3.95) combine to $-i\mathcal{I}\delta_1\bar{\psi}\bar{\sigma}^\mu\psi$; the third and fourth terms in (3.94) combine with the second and third terms in (3.95) respectively. The final expression for $V_1^{II\mu}$ is then given by

$$
\begin{aligned}
V_1^{II\mu} \quad = \quad & \delta_1 A{\pi_A^{II}}^\mu - i\mathcal{I}\delta_1\bar{\psi}\bar{\sigma}^\mu\psi + \delta_1\psi{\pi_\psi^{II}}^\mu + \delta_1\lambda{\pi_\lambda^{II}}^\mu \\
& + \frac{1}{2i}\mathcal{F}^{\dagger''}\epsilon_1\sigma_\nu\bar{\lambda}v^{*\mu\nu} + \frac{1}{2\sqrt{2}}\mathcal{F}^{\dagger'''}\epsilon_1\sigma^\mu\bar{\psi}\bar{\lambda}^2 \qquad (3.97)
\end{aligned}
$$

The rigid current is

$$
{N_1^{II}}^\mu = \delta_1 A{\pi_A^{II}}^\mu + \delta_1 v_\nu\Pi^{\mu\nu} + \delta_1\psi{\pi_\psi^{II}}^\mu + \delta_1\lambda{\pi_\lambda^{II}}^\mu \qquad (3.98)
$$

thus the total current is given by

$$
J_1^{II\mu} = i\mathcal{I}\delta_1\bar{\psi}\bar{\sigma}^\mu\psi - \frac{1}{2i}\mathcal{F}''^\dagger\epsilon_1\sigma_\nu\bar{\lambda}v^{*\mu\nu} - \frac{1}{2\sqrt{2}}\mathcal{F}'''^\dagger\epsilon_1\sigma^\mu\bar{\psi}\bar{\lambda}^2 + \delta_1 v_\nu\Pi^{\mu\nu} \quad (3.99)
$$

Again we see that writing explicitly the momenta and the transformations we recover the expression (3.78).
We conclude that the current is once and for all given by (3.78), but in order to implement the canonical procedure we have to express that current *either* as in (3.93) *or* as in (3.99) and stick to it.

We can now impose the temporal gauge for the vector field $v^0 = 0$ and introduce the canonical conjugate momenta of the fields. As in the classical case we define $\pi_{\text{any field}} \equiv \pi^0_{\text{any field}}$. Thus it is simply matter to pick up the time component of (3.85) and (3.86), for the Lagrangian $\mathcal{L}^I$, or (3.87) and (3.88), for the Lagrangian $\mathcal{L}^{II}$. Note that for $\Pi^i \equiv \Pi^{0i}$ there is no difference, it is always given by the time component of (3.68).
The basic non zero equal time Poisson brackets are the same as in the classical case, namely

$$\{A(x), \pi_A(y)\}_- = \{A^\dagger(x), \pi_{A^\dagger}(y)\}_- = \delta^{(3)}(\vec{x} - \vec{y}) \tag{3.100}$$

$$\{v_i(x), \Pi^j(y)\}_- = \delta_i^j \delta^{(3)}(\vec{x} - \vec{y}) \tag{3.101}$$

and[5]

$$\begin{aligned} \{\bar\psi_{\dot\alpha}(x)\ ,\ (\pi^I_{\bar\psi})^{\dot\beta}(y)\}_+ &= \delta_{\dot\alpha}^{\dot\beta}\delta^{(3)}(\vec{x} - \vec{y}) \\ \{\bar\lambda_{\dot\alpha}(x)\ ,\ (\pi^I_{\bar\lambda})^{\dot\beta}(y)\}_+ &= \delta_{\dot\alpha}^{\dot\beta}\delta^{(3)}(\vec{x} - \vec{y}) \end{aligned} \tag{3.102}$$

or

$$\begin{aligned} \{\psi_{\alpha}(x)\ ,\ (\pi^{II}_{\psi})^{\beta}(y)\}_+ &= \delta_{\alpha}^{\beta}\delta^{(3)}(\vec{x} - \vec{y}) \\ \{\lambda_{\alpha}(x)\ ,\ (\pi^{II}_{\lambda})^{\beta}(y)\}_+ &= \delta_{\alpha}^{\beta}\delta^{(3)}(\vec{x} - \vec{y}) \end{aligned} \tag{3.103}$$

[5]See also Appendix B. For instance, there we explain the conventions for $\{\bar\psi^{\dot\alpha}\ ,\ (\pi_{\bar\psi})^{\dot\beta}\}_+ = \{(\pi_{\bar\psi})^{\dot\beta}\ ,\ \bar\psi^{\dot\alpha}\}_+ = \epsilon^{\dot\alpha\dot\beta}\delta^{(3)}(\vec{x} - \vec{y})$.

but, many subtleties have to be handled with care. Classically there is no effect of the partial integration on the bosonic momenta. Effectively this is no longer the case, as we have seen, but their Poisson brackets do not depend on which Lagrangian one uses ($\mathcal{L}^I$ or $\mathcal{L}^{II}$), thus we did not write an index I or II on the momenta.

On the other hand, the fermionic brackets, classically and effectively, do depend on the Lagrangian used. Nevertheless we could easily derive a formula which does not depend on the partial integration. At this end we have simply to notice that the expression of the conjugate momenta in the two settings, $(\pi^I_{\bar\psi})_{\dot\alpha} = i\mathcal{I}\psi^\alpha\sigma^0_{\alpha\dot\alpha}$ and $(\pi^{II}_\psi)^\alpha = i\mathcal{I}\bar\psi_{\dot\alpha}\bar\sigma^{0\dot\alpha\alpha}$ (same for λ) implies that the canonical commutations (3.102) and (3.103) are both equivalent to

$$\{\psi_\alpha(x)\ ,\ \bar\psi_{\dot\alpha}(y)\}_+ = -\frac{i}{\mathcal{I}}\sigma^0_{\alpha\dot\alpha}\delta^{(3)}(\vec x - \vec y) \tag{3.104}$$

(same for λ). Thus we can use either (3.104) or one of the two canonical brackets (3.102) and (3.103).

The other Poisson brackets are all zero. Note for instance that

$$\{\Pi^i\ ,\ \chi\}_- = 0 \tag{3.105}$$

where $\chi \equiv \psi, \lambda, \bar\psi, \bar\lambda$, even if the effective Π^i has all the fermions. Furthermore, it is crucial to notice that the usual assumption that the Poisson brackets of bosons and fermions are all zero no longer holds. Choosing for instance the first setting, we must have $\{\pi_A\ ,\ \bar\psi\}_- = \{\pi_{A^\dagger}\ ,\ \bar\psi\}_- = 0$. If we take into account that

$$\{\pi_A\ ,\ \mathcal{I}\}_- = -\frac{1}{2i}\mathcal{F}''' \quad \{\pi_{A^\dagger}\ ,\ \mathcal{I}\}_- = \frac{1}{2i}\mathcal{F}'''^\dagger \tag{3.106}$$

and the above mentioned definition of $\pi^I_{\bar\psi}$ we also have

$$\{\pi_A\ ,\ \pi_{\bar\psi}\}_- = 0 \ \Rightarrow\ \{\pi_A\ ,\ \psi\}_- = -\frac{i}{2}f\psi \tag{3.107}$$

$$\{\pi_{A^\dagger}\ ,\ \pi_{\bar\psi}\}_- = 0 \ \Rightarrow\ \{\pi_{A^\dagger}\ ,\ \psi\}_- = +\frac{i}{2}f^\dagger\psi \tag{3.108}$$

where $f = \mathcal{F}'''/\mathcal{I}$. Similar formulae hold for λ.

3.3.3 Verification that the Q_α's generate the Susy transformations

At this point really we have to verify that our charges produce the given Susy transformations. As explained in the first Chapter, this is a very delicate point for Susy and, more generally, for any space-time symmetry. In the simple case of the classical theory we succeeded in doing that, but for the highly non trivial effective theory we have to be more careful. For instance the charges $\hat{Q}_\alpha$ obtained by letting the Susy parameters become local (see Eq.(1.15)) do not work in this sense, as can be seen in [19].

In the following Section we shall choose the setting I to compute the centre Z. For the moment we want to show how in both cases our charges produce the Susy transformations.

If we choose the current (3.93), the charge is given by

$$\epsilon_1 Q_1^I = \int d^3x[\delta_1\bar{\psi}\pi^I_{\bar{\psi}} + \Pi^i\delta_1 v_i - \frac{1}{2i}\mathcal{F}^{\dagger\prime\prime}\epsilon_1\sigma_i\bar{\lambda}v^{*0i} - \frac{1}{2\sqrt{2}}\mathcal{F}^{\dagger\prime\prime\prime}\epsilon_1\sigma^0\bar{\psi}\bar{\lambda}^2] \quad (3.109)$$

Using the same conventions as for the classical case, we shall call Δ_1 the transformation induced by this charge. Again we have to conveniently express it in terms of fermionic and bosonic variables.

We have:

$$\begin{aligned}\Delta_1 A(x) \equiv \{A(x)\ ,\ \epsilon_1 Q_1^I\}_- &= \int d^3y\{A(x)\ ,\ \delta_1\bar{\psi}(y)\pi^I_{\bar{\psi}}(y)\}_- \\ &= \int d^3y\{A(x)\ ,\ \sqrt{2}\epsilon_1\sigma^\nu\bar{\sigma}^0\psi(y)\mathcal{I}(y)\partial_\nu A^\dagger(y)\}_- \\ &= \int d^3y\{A(x)\ ,\ \sqrt{2}\epsilon_1\psi(y)\pi^I_A(y) + \text{irr.}\}_- \\ &= \sqrt{2}\epsilon_1\psi(x) = \delta_1 A(x) \qquad (3.110)\end{aligned}$$

where "irr." stands for terms irrelevant for the Poisson bracket.

$$\Delta_1 A^\dagger(x) \equiv \{A^\dagger(x)\ ,\ \epsilon_1 Q_1^I\}_- = 0 = \delta_1 A^\dagger(x) \tag{3.111}$$

$$\Delta_1 v_i(x) \equiv \{v_i(x)\ ,\ \epsilon_1 Q_1^I\}_- = \int d^3y \{v_i(x)\ ,\ \Pi^j(y)\}_- \delta_1 v_j(y) = \delta_1 v_i(x) \tag{3.112}$$

$$\begin{aligned}\Delta_1 \bar\psi_{\dot\alpha}(x) \equiv \{\epsilon_1 Q_1^I\ ,\ \bar\psi_{\dot\alpha}(x)\}_- &= \int d^3y \delta_1 \bar\psi_{\dot\beta}(y) \{\pi_{\bar\psi}^{I\dot\beta}(y)\ ,\ \bar\psi_{\dot\alpha}(x)\}_+ \\ &= \delta_1 \bar\psi_{\dot\alpha}(x)\end{aligned} \tag{3.113}$$

$$\Delta_1 \bar\lambda_{\dot\alpha}(x) \equiv \{\epsilon_1 Q_1^I\ ,\ \bar\lambda_{\dot\alpha}(x)\}_- = 0 = \delta_1 \bar\lambda_{\dot\alpha}(x) \tag{3.114}$$

For $\Delta_1 \pi^I_{\bar\psi\dot\alpha}$ some attention is due to the fact that $\pi^I_{\bar\psi\dot\alpha}$ is a product of a bosonic function $\mathcal{I}$ and of a fermion ψ. On the one hand

$$\begin{aligned}\Delta_1 \pi^I_{\bar\psi\dot\alpha}(x) \equiv \{\epsilon_1 Q_1^I\ ,\ \pi^I_{\bar\psi\dot\alpha}(x)\}_- &= \int d^3y (-\frac{1}{2\sqrt2}\mathcal{F}'''^\dagger \{\epsilon_1 \sigma^0 \bar\psi\ ,\ \pi^I_{\bar\psi\dot\alpha}(x)\}_- \bar\lambda^2) \\ &= -\frac{1}{2\sqrt2}\mathcal{F}'''^\dagger \epsilon_1^\alpha \sigma^0_{\alpha\dot\alpha} \bar\lambda^2\end{aligned} \tag{3.115}$$

on the other hand, writing explicitly $\pi^I_{\bar\psi\dot\alpha}$ we have

$$\Delta_1 \pi^I_{\bar\psi\dot\alpha}(x) = \frac{1}{\sqrt2}\mathcal{F}''' \epsilon_1 \psi \psi^\alpha \sigma^0_{\alpha\dot\alpha} + i\mathcal{I}\sigma^0_{\alpha\dot\alpha}\Delta_1\psi^\alpha \tag{3.116}$$

where we have used $\Delta_1 \mathcal{I} = \frac{1}{2i}(\mathcal{F}'''\Delta_1 A - \mathcal{F}^{\dagger'''}\Delta_1 A^\dagger)$. Thus, by comparing the two expressions for $\Delta_1 \pi^I_{\bar\psi\dot\alpha}$ we obtain

$$\Delta_1 \psi^\alpha = \sqrt2 \epsilon_1^\alpha (-\frac{i}{4} f \psi^2 + \frac{i}{4} f^\dagger \bar\lambda^2) = \sqrt2 \epsilon_1^\alpha E_{\text{on}} = \delta_1^{\text{on}} \psi^\alpha \tag{3.117}$$

where we have used the expression (3.73) for E on-shell and the Fierz identity $\psi_\alpha \psi^\beta = -\frac12 \delta_\alpha^\beta \psi^2$.

More labour is needed to compute $\Delta_1 \lambda$ from $\Delta_1 \pi^I_{\bar\lambda\dot\alpha}$ and we leave this to Appendix E. The result of that computation is the following

$$\Delta_1 \lambda^\beta = -\epsilon_1^\alpha (\sigma^{\mu\nu})_\alpha^\beta v_{\mu\nu} + i\epsilon_1^\beta (-\frac{1}{2\sqrt2}(f\psi\lambda + f^\dagger \bar\psi\bar\lambda)) = \delta_1^{\text{on}} \lambda^\beta \tag{3.118}$$

where we have used again the expression of the dummy fields on shell (3.71). We can conclude that in the effective case as well $\Delta_1 \equiv \delta_1^{\rm on}$. Thus our effective charge $\epsilon_1 Q_1^I$ correctly generates the first supersymmetry transformations (3.2)-(3.4). The second set of supersymmetry transformations (3.5)-(3.7) is obtained by first replacing the charge in (3.109) by its R-symmetric counterpart[6]

$$\epsilon_2 Q_2^I = \int d^3x [\delta_2 \bar{\lambda} \pi^I_{\bar{\lambda}} + \Pi^i \delta_2 v_i - \frac{1}{2i} \mathcal{F}^{\dagger\prime\prime} \epsilon_2 \sigma_i \bar{\psi} v^{*0i} + \frac{1}{2\sqrt{2}} \mathcal{F}^{\dagger\prime\prime\prime} \epsilon_2 \sigma^0 \bar{\lambda} \bar{\psi}^2] \quad (3.119)$$

and then performing for Δ_2 the same kind of computations we have done so far for Δ_1. We immediately see that this charge reproduces the correct transformations for A ("R mirror" computation of (3.110)), $A^\dagger$ ($\pi^I_{A^\dagger}$ is absent), v_i (trivial), $\bar{\lambda}$ (trivial) and $\bar{\psi}$ ($\pi^I_{\bar{\psi}}$ absent). By a direct "R mirror" check we also reconstruct the transformations for λ and ψ. We give in Appendix E the explicit computation for the tricky one, $\Delta_2 \psi$.

We also leave to Appendix E the interesting check that Q_1^{II} as well generates the transformations (3.2)-(3.4). As we shall show there, in this case the delicate point is to handle the $\mathcal{F}'''$ term in π_A^{II}. This problem is absent for $\epsilon_1 Q_1^I$ due to the fact that $\pi_A^I = \pi_A$ and there is no $\pi^I_{A^\dagger}$ in the charge, as we expect being $\delta_1 A^\dagger = 0$. Note that even if we use $\epsilon_1 Q_1^I$ the same problem will appear in handling $\bar{\epsilon}_1 \bar{Q}_1^I$ where we have $\pi^I_{A^\dagger}$. This means that we have to express the time derivative of the scalar field in terms of the correspondent canonical momentum and commute this expression with the fields and momenta. From this follows that it is simpler to compute the central charge Z with the Q^I's, whereas for the Hamiltonian there is no such a computational advantage. Thus we shall choose the "$\mathcal{L}^I$ setting" for

[6]Note that under R-symmetry $\Pi^i \to -\Pi^i$ due to $v^i \to -v^i$ and $\lambda \sigma^{0i} \psi \to +\psi \sigma^{0i} \lambda = -\lambda \sigma^{0i} \psi$.

our computations, being now sure that the results will be the same in the other setting.

3.3.4 The central charge

Another interesting check is the computation of the Hamiltonian H. Since we shall perform this computation in the SU(2) sector we do not show it here. The main point is the computation of the central charge. Now it is an easy matter.

Let us first write the R-symmetric of (3.109) given by

$$\epsilon_2 Q_2 = \int d^3x \Big(\Pi^j \delta_2 v_j + \delta_2 \bar{\lambda} \pi_{\bar{\lambda}} + \frac{i}{2} \mathcal{F}^{\dagger\prime\prime} \epsilon_2 \sigma_j \bar{\psi} v^{*0j} + \frac{1}{2\sqrt{2}} \mathcal{F}^{\dagger\prime\prime\prime} \epsilon_2 \sigma^0 \bar{\lambda} \bar{\psi}^2 \Big) \tag{3.120}$$

We have simply to commute the two charges as we did in the classical case, paying due attention to the subtleties discussed earlier. The eight terms (four pairs) different from zero are

$$\begin{aligned}
\{\epsilon_1 Q_1, \epsilon_2 Q_2\}_- \;=\; & \int d^3x d^3y \Big(\{\Pi^i, v^{*0j}\}_- \delta_1 v_i \frac{i}{2} \mathcal{F}^{\dagger\prime\prime} \epsilon_2 \sigma_j \bar{\psi} \\
& + \{v^{*0i}, \Pi^j\}_- \delta_2 v_j \frac{i}{2} \mathcal{F}^{\dagger\prime\prime} \epsilon_1 \sigma_i \bar{\lambda} && (3.121) \\
& + \delta_1 \bar{\psi}_{\dot{\alpha}} \{\pi^{\dot{\alpha}}_{\bar{\psi}}, \bar{\psi}^2\}_- \frac{1}{2\sqrt{2}} \mathcal{F}^{\dagger\prime\prime\prime} \epsilon_2 \sigma^0 \bar{\lambda} \\
& - \delta_2 \bar{\lambda}_{\dot{\alpha}} \{\bar{\lambda}^2, \pi^{\dot{\alpha}}_{\bar{\lambda}}\}_- \frac{1}{2\sqrt{2}} \mathcal{F}^{\dagger\prime\prime\prime} \epsilon_1 \sigma^0 \bar{\psi} && (3.122) \\
& + \Pi^i \delta_2 \bar{\lambda}_{\dot{\alpha}} \{\delta_1 v_i, \pi^{\dot{\alpha}}_{\bar{\lambda}}\}_- \\
& + \Pi^j \delta_1 \bar{\psi}_{\dot{\alpha}} \{\pi^{\dot{\alpha}}_{\bar{\psi}}, \delta_2 v_j\}_- && (3.123) \\
& + \delta_1 \bar{\psi}_{\dot{\alpha}} \{\pi^{\dot{\alpha}}_{\bar{\psi}}, \epsilon_2 \sigma_j \bar{\psi}\}_- \frac{i}{2} \mathcal{F}^{\dagger\prime\prime} v^{*0j} \\
& + \delta_2 \bar{\lambda}_{\dot{\alpha}} \{\epsilon_1 \sigma_i \bar{\lambda}, \pi^{\dot{\alpha}}_{\bar{\lambda}}\}_- \frac{i}{2} \mathcal{F}^{\dagger\prime\prime} v^{*0i} \Big) && (3.124)
\end{aligned}$$

The terms (3.121) combine to a term in any respect similar to the classical counterpart (3.57). It is of the form $\mathcal{F}^{\dagger\prime\prime} \partial(\bar{\psi}\bar{\lambda})$. When we write explicitly

$\delta_1\bar{\psi} = -i\sqrt{2}\epsilon_1 \not{\partial} A^\dagger$ and $\delta_2\bar{\lambda} = i\sqrt{2}\epsilon_2 \not{\partial} A^\dagger$ we see that the terms (3.122) combine to a term of the form $(\partial\mathcal{F}^{\dagger''})\bar{\psi}\bar{\lambda}$. Thus from these terms we obtain the total divergence given by

$$\int d^3x\partial_i[i\mathcal{F}^{\dagger''}(\epsilon_1\sigma^0\bar{\psi}\epsilon_2\sigma^i\bar{\lambda} - \epsilon_2\sigma^0\bar{\lambda}\epsilon_1\sigma^i\bar{\psi})] \tag{3.125}$$

Again by explicitly writing $\delta_1\bar{\psi}$ and $\delta_2\bar{\lambda}$, we see tha the terms (3.123) and (3.124) give a total divergence and two additional terms, as in the classical case

$$\int d^3x\Big(\partial_i[2\sqrt{2}\Pi^i A^\dagger + \sqrt{2}v^{*0i}\mathcal{F}'^\dagger] + 2\sqrt{2}(\partial_i\Pi^i)A^\dagger + \sqrt{2}(\partial_i v^{*0i})\mathcal{F}'^\dagger\Big)\epsilon_1\epsilon_2 \tag{3.126}$$

Imposing the Bianchi identities and the Gauss law, dropping the Susy parameters ϵ_1 and ϵ_2, using the formula $Z = \frac{i}{4}\epsilon^{\alpha\beta}\{Q_{1\alpha}, Q_{2\beta}\}_+$, reintroducing the factor 4π and dropping the fermionic term as in the classical case, we can write

$$Z = i\sqrt{2}\int d^2\vec{\Sigma}\cdot(\vec{\Pi}A^\dagger + \frac{1}{4\pi}\vec{B}A_D^\dagger) \tag{3.127}$$

where $d^2\vec{\Sigma}$ is the measure on the sphere at infinity S^2_∞, $B^i = \frac{1}{2}\epsilon^{0ijk}v_{jk}$ as in the classical case, and we introduced the SW dual of the scalar field $A^\dagger$

$$A_D^\dagger \equiv \mathcal{F}'^\dagger(A^\dagger) \tag{3.128}$$

Surprisingly enough the expression (3.127) is *formally* identical to the classical one given in (3.64). We see that the topological nature of Z is sufficient to protect its form at the quantum level. All one has to do is to use a little dictionary and replace classical quantities with their quantum counterparts. Thus we can apply exactly the same logic as in the classical case and define the electric and magnetic charges *à la* Witten and Olive. The final expression is

$$Z = i\sqrt{2}(n_e a^* + n_m a_D^*) \tag{3.129}$$

where $< 0|A^\dagger|0 >= a^*$, $< 0|A_D^\dagger|0 >= a_D^*$ and n_e, n_m are the electric and magnetic quantum numbers, respectively.

Eventually we proved the SW mass formula. At this end we can simply use the BPS type of argument given in [3] or [43], noticing that our direct computation includes fermions but they occur as a total divergence which falls off fast enough to give contribution on S^2_∞. Thus

$$M = |Z| = \sqrt{2}|n_e a + n_m \mathcal{F}'(a)| \tag{3.130}$$

A last remark is now in order. The U(1) low energy theory is invariant under the linear shift $\mathcal{F}(A) \to \mathcal{F}(A) + cA$. This produces an ambiguity in the definition of Z. For this and other purposes we want also to analyse the SU(2) high energy theory in the next Chapter.

Chapter 4

SW SU(2) High Energy Sector

We want now to generalize the results obtained in the previous Chapter to the high energy sector taking into account all the effective SU(2) fields, massive and massless.

We intend to clarify the following points. First, the U(1) Lagrangian is invariant under the linear shift $\mathcal{F}(A) \to \mathcal{F}(A) + cA$, where c is a c-number. In principle, this induces an ambiguity in the central charge due to the presence of $\mathcal{F}'(A)$. This ambiguity can be removed in the full high energy theory, where the prepotential is a function of the SU(2) Casimir $A^a A^a$, and such a linear shift is not allowed, since it would break the SU(2) gauge symmetry. Second, we want to see what is the role in the mass formula of the heavy fields. Third, the SU(2) theory has non trivial features, absent in the low energy sector, as for instance, a non trivial Gauss law. We want to test our Susy Noether recipe on this more complicated ground as well, even if we do not expect any change with respect to the U(1) case.

In the first Section we shall construct a unique charge $Q_{1\alpha}$ for the first Susy starting from its U(1) limit. In the second Section we shall commute this charge with its complex conjugate to obtain the Hamiltonian and, by Legendre transforming it, the Lagrangian. The Chapter ends with the computation of the central charge Z by commuting $Q_{1\alpha}$ with its R-symmetric counterpart.

4.1 The SU(2) Susy charges

To construct the SU(2) Susy charges[1] we shall write the most natural generalization of the U(1) Susy charges obtained in the last Chapter, impose canonicity and define the SU(2) fields and conjugate momenta and finally fix them by requiring that they generate the given Susy transformations.

Before starting our journey let us introduce the SU(2) notation we shall use and make few remarks.

A generic SU(2) vector is defined as $\vec{X} = \frac{1}{2}\sigma^a X^a$ with $a = 1, 2, 3$ and we follow the summation convention. The σ^a's are the standard Pauli matrices satisfying $[\sigma^a, \sigma^b] = 2i\epsilon^{abc}\sigma^c$, where ϵ^{abc} are the structure constants of SU(2), and $\mathrm{Tr}\sigma^a\sigma^b = 2\delta^{ab}$. The covariant derivative and the vector field strength are given by $\mathcal{D}_\mu\vec{X} = \partial_\mu\vec{X} - i[\vec{v}_\mu, \vec{X}]$ and $\vec{v}_{\mu\nu} = \partial_\mu\vec{v}_\nu - \partial_\nu\vec{v}_\mu - i[\vec{v}_\mu, \vec{v}_\nu]$, respectively.

We shall work in components thus it is convenient to write down these formulae explicitly

$$\mathcal{D}_\mu X^a = \partial_\mu X^a + \epsilon^{abc} v_\mu^b X^c \tag{4.1}$$

and

$$v_{\mu\nu}^a = \partial_\mu v_\nu^a - \partial_\nu v_\mu^a + \epsilon^{abc} v_\mu^b v_\nu^c \tag{4.2}$$

[1] As in the Abelian case we can concentrate on the first Susy charge.

Some authors, [26], [46], [21], keep the renormalizable SU(2) gauge coupling g even in the effective theory (for instance their covariant derivatives are defined as $\mathcal{D}_\mu X^a = \partial_\mu X^a + g\epsilon^{abc} v^b_\mu X^c$). This is somehow misleading since, as discussed earlier, in SW theory the effective coupling is once and for all given by $\tau(a) = \mathcal{F}''(a)$. Of course the microscopic theory is scale invariant before SSB[2], and a redefinition of the fields $gX \to X$ does no harm. The matter is less clear in the effective theory, where even the definition of what is a field poses some problems and scale invariance is lost after SSB. Therefore we prefer to follow the conventions of Seiberg and Witten [3], where already at microscopic level the g is absorbed in the definition of the fields and only appears in the overall factor $1/g^2$ (see also our expression for the U(1) classical Lagrangian in (3.14)).

Nevertheless we can keep track of g since by charge conjugation[3] $g \to -g$ (see for instance [48]), which in our notation becomes $\epsilon^{abc} \to -\epsilon^{abc}$.

Finally, as we have seen in Chapter 2, the SU(2) prepotential $\mathcal{F}$ is a holomorphic function of the SU(2) gauge Casimir $A^a A^a$, $a = 1, 2, 3$. Our $\mathcal{F}$ corresponds to the function $\mathcal{H}$ in Seiberg and Witten conventions [3]. For some properties of this function see also Appendix F.

Let us now write down the SU(2) generalization [46] of the U(1) Susy transformations given in (3.2)-(3.7)

first supersymmetry, parameter ϵ_1

[2]As a matter of fact, it is invariant under the full superconformal group.

[3]In the Abelian case we implemented the R-symmetry as $\psi \leftrightarrow -\lambda$, $E \leftrightarrow E^\dagger$ and $v_\mu \to -v_\mu$ (charge conjugation) when $\epsilon_1 \to \epsilon_2$. Noticing that the doublet $\begin{pmatrix} \psi \\ \lambda \end{pmatrix}$ transforms under $-\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{U}(2)$ we can say that there is room for the charge conjugation $v_\mu \to -v_\mu$ which becomes the $\mathbf{Z}_2$ discrete part of U(2).

$$\begin{aligned}\delta_1\vec{A} &= \sqrt{2}\epsilon_1\vec{\psi}\\ \delta_1\vec{\psi}^\alpha &= \sqrt{2}\epsilon_1^\alpha\vec{E}\\ \delta_1\vec{E} &= 0\end{aligned}\tag{4.3}$$

$$\begin{aligned}\delta_1\vec{E}^\dagger &= i\sqrt{2}\epsilon_1\ \not{\mathcal{D}}\vec{\bar{\psi}}+2i[\vec{A}^\dagger,\epsilon_1\vec{\lambda}]\\ \delta_1\vec{\bar{\psi}}_{\dot\alpha} &= -i\sqrt{2}\epsilon_1^\alpha\ \mathcal{D}_{\alpha\dot\alpha}\vec{A}^\dagger\\ \delta_1\vec{A}^\dagger &= 0\end{aligned}\tag{4.4}$$

$$\begin{aligned}\delta_1\vec{\lambda}^\alpha &= -\epsilon_1^\beta(\sigma^{\mu\nu}{}_\beta{}^\alpha\vec{v}_{\mu\nu}-i\delta_\beta^\alpha\vec{D})\\ \delta_1\vec{v}^\mu &= i\epsilon_1\sigma^\mu\vec{\bar{\lambda}}\quad \delta_1\vec{D}=-\epsilon_1\ \not{\mathcal{D}}\vec{\bar{\lambda}}\\ \delta_1\vec{\bar{\lambda}}_{\dot\alpha} &= 0\end{aligned}\tag{4.5}$$

second supersymmetry, parameter ϵ_2

$$\begin{aligned}\delta_2\vec{A} &= -\sqrt{2}\epsilon_2\vec{\lambda}\\ \delta_2\vec{\lambda}^\alpha &= -\sqrt{2}\epsilon_2^\alpha\vec{E}^\dagger\\ \delta_2\vec{E}^\dagger &= 0\end{aligned}\tag{4.6}$$

$$\begin{aligned}\delta_2\vec{E} &= -i\sqrt{2}\epsilon_2\ \not{\mathcal{D}}\vec{\bar{\lambda}}+2i[\vec{A}^\dagger,\epsilon_2\vec{\psi}]\\ \delta_2\vec{\bar{\lambda}}_{\dot\alpha} &= i\sqrt{2}\epsilon_2^\alpha\ \mathcal{D}_{\alpha\dot\alpha}\vec{A}^\dagger\\ \delta_2\vec{A}^\dagger &= 0\end{aligned}\tag{4.7}$$

$$\begin{aligned}\delta_2\vec{\psi}^\alpha &= -\epsilon_2^\beta(\sigma^{\mu\nu}{}_\beta{}^\alpha\vec{v}_{\mu\nu}+i\delta_\beta^\alpha\vec{D})\\ \delta_2\vec{v}^\mu &= i\epsilon_2\sigma^\mu\vec{\bar{\psi}}\quad \delta_2\vec{D}=\epsilon_2\ \not{\mathcal{D}}\vec{\bar{\psi}}\\ \delta_2\vec{\bar{\psi}}_{\dot\alpha} &= 0\end{aligned}\tag{4.8}$$

We want now to construct the SU(2) Susy charges that generate these transformations by generalizing the U(1) Susy charges obtained in the last Chapter. At this end we simply write down the explicit expression of the first U(1) charge as the spatial integral of the time component of the explicit current J_1^μ given in (3.78)

$$\begin{aligned}\epsilon_1 Q_1^{\mathrm{U}(1)} &= \int d^3x \Big(\sqrt{2}\mathcal{I}\epsilon_1(\partial\!\!\!/ A^\dagger)\bar{\sigma}^0\psi - \frac{1}{2i}\mathcal{F}^{\dagger\prime\prime}\epsilon_1\sigma_i\bar{\lambda}v^{*0i} - \frac{1}{2\sqrt{2}}\mathcal{F}^{\dagger\prime\prime\prime}\epsilon_1\sigma^0\bar{\psi}\bar{\lambda}^2 \\ &+ [-\frac{1}{2}(\mathcal{F}''\hat{v}^{0i} - \mathcal{F}^{\dagger\prime\prime}\hat{v}^{\dagger 0i}) + \frac{1}{\sqrt{2}}(\mathcal{F}'''\lambda\sigma^{0i}\psi - \mathcal{F}^{\dagger\prime\prime\prime}\bar{\lambda}\bar{\sigma}^{0i}\bar{\psi})]\epsilon_1\sigma_i\bar{\lambda}\Big)\end{aligned} \tag{4.9}$$

where, for the moment, we scale $\mathcal{F}$ by a factor of 4π.

Then we generalize this expression in the most natural way, namely

$$\begin{aligned}\epsilon_1 Q_1^{\mathrm{SU}(2)} \equiv& \int d^3x \Big(\sqrt{2}\mathcal{I}^{ab}\epsilon_1(\partial\!\!\!/ A^{a\dagger})\bar{\sigma}^0\psi^b - \frac{1}{2i}\mathcal{F}^{ab\dagger}\epsilon_1\sigma_i\bar{\lambda}^a v^{*b0i} \\ &- \frac{1}{2\sqrt{2}}\mathcal{F}^{abc\dagger}\epsilon_1\sigma^0\bar{\psi}^a\bar{\lambda}^b\bar{\lambda}^c + [-\frac{1}{2}(\mathcal{F}^{ab}\hat{v}^{a0i} - \mathcal{F}^{ab\dagger}\hat{v}^{\dagger a0i}) \\ &+ \frac{1}{\sqrt{2}}(\mathcal{F}^{bcd}\lambda^c\sigma^{0i}\psi^d - \mathcal{F}^{bcd\dagger}\bar{\lambda}^c\bar{\sigma}^{0i}\bar{\psi}^d)]\epsilon_1\sigma_i\bar{\lambda}^b\Big)\end{aligned} \tag{4.10}$$

where $\mathcal{F}^{a_1\cdots a_n} = \partial^n\mathcal{F}/\partial A^{a_1}\cdots\partial A^{a_n}$ (see Appendix F) and $\mathcal{F}^{ab} = \mathcal{R}^{ab} + i\mathcal{I}^{ab}$. In order to impose a canonical form to this charge we have to define the conjugate momenta of the fields and therefore produce the transformations above given. The SU(2) *version* of the U(1) conjugate momenta is given by

$$\Pi^{ai} = -\frac{1}{2i}(\mathcal{F}^{ab}\hat{v}^{0ib} - \mathcal{F}^{ab\dagger}\hat{v}^{\dagger 0ib}) + \frac{1}{i\sqrt{2}}\mathcal{F}^{abc}\lambda^b\sigma^{0i}\psi^c - \frac{1}{i\sqrt{2}}\mathcal{F}^{abc\dagger}\bar{\lambda}^b\bar{\sigma}^{0i}\bar{\psi}^c \tag{4.11}$$

$$\pi^a_A = -\mathcal{I}^{ab}\partial^0 A^{\dagger b} \qquad \pi^a_{A^\dagger} = -\mathcal{I}^{ab}\partial^0 A^b - \frac{1}{2}\mathcal{F}^{\dagger abc}(\bar{\psi}^b\bar{\sigma}^0\psi^c + \bar{\lambda}^b\bar{\sigma}^0\lambda^c) \tag{4.12}$$

$$\pi^a_{\psi} = i\mathcal{I}^{ab}\psi^b\sigma^0 \qquad \pi^a_{\lambda} = i\mathcal{I}^{ab}\lambda^b\sigma^0 \tag{4.13}$$

where we chose the *setting I* (see the correspondent U(1) expressions given in the previous Chapter in (3.85), (3.86) and (3.68)) and the temporal gauge for the vector field (thus $\mathcal{D}^0 = \partial^0$).

With these definitions the charge (4.10) becomes

$$\epsilon_1 Q_1^{\mathrm{SU}(2)} = \int d^3x \Big(\Pi^{ai} \delta_1 v_i^a + \delta_1 \bar{\psi}^a \pi_{\bar{\psi}}^a + \frac{i}{2} \mathcal{F}^{\dagger ab} \epsilon_1 \sigma_i \bar{\lambda}^a v^{*0ib} - \frac{1}{2\sqrt{2}} \mathcal{F}^{\dagger abc} \epsilon_1 \sigma^0 \bar{\psi}^a \bar{\lambda}^b \bar{\lambda}^c \Big) \tag{4.14}$$

By using the same techniques as in the Abelian case, we see that this charge correctly generates the transformations that do not involve dummy fields, namely

$$\delta_1 v_i^a \quad \delta_1 A^a \quad \delta_1 A^{\dagger a} \quad \delta_1 \bar{\psi}^a \quad \delta_1 \bar{\lambda}^a \tag{4.15}$$

To produce the other transformations one needs the explicit expressions of the dummy fields on shell. At this point we notice that one of the main differences between the U(1) and the SU(2) theories relies on the coupling of the dummy fields D^a to the scalar fields to give the Higgs potential, after elimination [14]. This potential (and the Yukawa potential) can never be reproduced in the Hamiltonian by commuting the charge (4.14) with its complex conjugate. Therefore we see that this first generalization needs to be improved.

We can obtain the missing terms by considering the classical (microscopic) SU(2) Lagrangian $\mathcal{L}_{\mathrm{class}}^{\mathrm{SU}(2)}$ and solving the Euler-Lagrange equations for D^a. The result is given by

$$(D^a)_{\mathrm{class}}^{\mathrm{on}} = i\epsilon^{abc} A^b A^{c\dagger} \tag{4.16}$$

where we used the standard expression for $\mathcal{L}_{\mathrm{class}}^{\mathrm{SU}(2)}$ (see, for instance, [21] and [46]). Note that $(E^a)_{\mathrm{class}}^{\mathrm{on}} = 0$. From the recipe given in the first Chapter and extensively applied in the U(1) case, we know that the charge has to produce the transformations with the dummy fields on shell. D^a appears in the transformation of λ^a therefore we want to produce $\delta_1^D \lambda^a$ from $\{\epsilon_1 Q_1^{\mathrm{SU}(2)}, \pi_{\bar{\lambda}}^a\}$, where $\pi_{\bar{\lambda}}^a = i\tau_I \lambda^a \sigma^0$ is the classical limit of the SU(2) effective conjugate momentum of $\bar{\lambda}^a$ given in (4.13). Thus we conclude that

a missing term in the classical charge is given by

$$i\tau_I \epsilon_1 \sigma^0 \bar{\lambda}^a \epsilon^{acd} A^c A^{d\dagger} \tag{4.17}$$

Furthermore this term is the only missing term, because once it is added then we obtain all the correct Susy transformations. The term in the effective charge that will produce (4.17) in the classical limit is evidently

$$i\mathcal{I}^{ab} \epsilon_1 \sigma^0 \bar{\lambda}^b \epsilon^{acd} A^c A^{d\dagger} \tag{4.18}$$

and it is therefore clear that we must add such a term to the charge in (4.14). We shall see in the next Section that, as in the classical theory, this term is the *only* new term that is required. It produces the Higgs and Yukawa potential in the Hamiltonian and it is responsible for most of the new terms in the centre.

Thus finally, the first Susy SU(2) effective charge is given by

$$\begin{aligned} \epsilon_1 Q_1 \quad &= \quad \int d^3x \Big(\Pi^{ai} \delta_1 v_i^a + \delta_1 \bar{\psi}^a \pi^a_{\bar{\psi}} + \frac{i}{2} \mathcal{F}^{\dagger ab} \epsilon_1 \sigma_i \bar{\lambda}^a v^{*0ib} \\ & \qquad - \frac{1}{2\sqrt{2}} \mathcal{F}^{\dagger abc} \epsilon_1 \sigma^0 \bar{\psi}^a \bar{\lambda}^b \bar{\lambda}^c + i\mathcal{I}^{ab} \epsilon_1 \sigma^0 \bar{\lambda}^b \epsilon^{acd} A^c A^{d\dagger} \Big) \end{aligned} \tag{4.19}$$

where we have dropped the label “SU(2)”.

We conclude this Section by writing the R-symmetric counterpart of (4.19), given by

$$\begin{aligned} \epsilon_2 Q_2 \quad &= \quad \int d^3x \Big(\Pi^{bj} \delta_2 v_j^b + \delta_2 \bar{\lambda}^b \pi^b_{\bar{\lambda}} + \frac{i}{2} \mathcal{F}^{\dagger cd} \epsilon_2 \sigma_j \bar{\psi}^c v^{*0jd} \\ & \qquad + \frac{1}{2\sqrt{2}} \mathcal{F}^{\dagger def} \epsilon_2 \sigma^0 \bar{\lambda}^d \bar{\psi}^e \bar{\psi}^f + i\mathcal{I}^{ef} \epsilon_2 \sigma^0 \bar{\psi}^f \epsilon^{egh} A^g A^{h\dagger} \Big) \end{aligned} \tag{4.20}$$

which generates the second set of Susy transformations above given, and the complex conjugate of (4.19), given by

$$\begin{aligned} \bar{\epsilon}_1 \bar{Q}_1 \quad &= \quad \int d^3x \Big(\Pi^{ai} \bar{\delta}_1 v_i^a + \sqrt{2} \mathcal{I}^{ab} \bar{\epsilon}_1 \; \bar{\mathcal{D}} A^a \sigma^0 \bar{\psi}^b + \frac{i}{2} \mathcal{F}^{ab} \bar{\epsilon}_1 \sigma_i \lambda^a v^{*0ib} \\ & \qquad + \frac{1}{2\sqrt{2}} \mathcal{F}^{abc} \bar{\epsilon}_1 \bar{\sigma}^0 \psi^a \lambda^b \lambda^c - \bar{\epsilon}_1 \pi^a_{\bar{\lambda}} \epsilon^{acd} A^c A^{d\dagger} \Big) \end{aligned} \tag{4.21}$$

where we introduced the conjugate momentum $\pi^a_{\bar{\lambda}}$.

4.2 Hamiltonian and Lagrangian

We are now in the position to compute the Hamiltonian H. The main points here are: first to compare the SU(2) Lagrangian obtained by Legendre transforming H with the U(1) Lagrangian and with the Lagrangian obtained by superfield expansion; second to obtain the non trivial Gauss law for the SU(2) theory. The last point is vital for our purpose, since our main interest is the computation of the central charge where the Gauss law is expected to play an important role.

We computed H by taking the Poisson brackets of $\epsilon_1 Q_1$ given in (4.19) with $\bar{\epsilon}_1 \bar{Q}_1$ given in (4.21), then getting rid of ϵ_1 and $\bar{\epsilon}_1$ ($\{\epsilon_1 Q_1\ ,\ \bar{\epsilon}_1 \bar{Q}_1\}_- = \epsilon_1^\alpha \bar{\epsilon}_1^{\dot{\alpha}} \{Q_{1\alpha}\ ,\ \bar{Q}_{1\dot{\alpha}}\}_+$) and finally taking the trace with $\bar{\sigma}^{0\dot{\alpha}\alpha}$. The final formula for H is

$$H = -\frac{i}{4} \bar{\sigma}^{0\dot{\alpha}\alpha} \{Q_{1\alpha}\ ,\ \bar{Q}_{1\dot{\alpha}}\}_+ \tag{4.22}$$

where we defined $H = P^0 = -P_0$. This lengthy computation is illustrated in some details in Appendix F. Its final result is given by

$$\begin{aligned}
H \;=\; & \int d^3x \Big(-\frac{1}{2} (\mathcal{I}^{ab})^{-1} \Pi^{ai} \Pi^{bi} - (\mathcal{I}^{ab})^{-1} \mathcal{R}^{bc} \Pi^{ai} B^{ic} - \frac{1}{2} (\mathcal{I}^{ab})^{-1} \mathcal{F}^{\dagger ab} \mathcal{F}^{ef} B^{ib} B^{if} \\
& - (\mathcal{I}^{ab})^{-1} \pi_A^a (\pi_A^b)^\dagger + \mathcal{I}^{ab} (\mathcal{D}^i A^a)(\mathcal{D}^i A^{\dagger b}) \\
& + i \mathcal{I}^{ab} \psi^a \sigma^i \mathcal{D}_i \bar{\psi}^b + i \mathcal{I}^{ab} \lambda^a \sigma^i \mathcal{D}_i \bar{\lambda}^b + \frac{1}{2} (\partial_i \mathcal{F}^{\dagger ab}) \bar{\lambda}^a \sigma^i \lambda^b \\
& - \frac{1}{\sqrt{2}} \mathcal{I}^{ad} \epsilon^{abc} (A^c \bar{\psi}^d \bar{\lambda}^b + A^{\dagger c} \psi^d \lambda^b) + \frac{1}{2} \mathcal{I}^{ab} \epsilon^{acd} \epsilon^{bfg} A^c A^{\dagger d} A^f A^{\dagger g} \\
& + \frac{i}{\sqrt{2}} (\mathcal{I}^{af})^{-1} \mathcal{F}^{feg} \psi^e \sigma_{i0} \lambda^g (\Pi^{ia} + \mathcal{F}^{\dagger ab} B^{ib}) \\
& - \frac{i}{\sqrt{2}} (\mathcal{I}^{ec})^{-1} \mathcal{F}^{\dagger abc} \bar{\psi}^a \bar{\sigma}_{i0} \bar{\lambda}^b (\Pi^{ie} + \mathcal{F}^{ed} B^{id}) \\
& + \frac{1}{16} \mathcal{F}^{\dagger efg} \mathcal{F}^{cad} (\mathcal{I}^{gc})^{-1} \bar{\psi}^e \bar{\psi}^f \psi^a \psi^c + \frac{1}{16} \mathcal{F}^{\dagger abc} \mathcal{F}^{efg} (\mathcal{I}^{ae})^{-1} \bar{\lambda}^b \bar{\lambda}^c \lambda^f \lambda^g \\
& + \frac{3}{16} \mathcal{F}^{\dagger bec} \mathcal{F}^{\dagger efg} (\mathcal{I}^{ab})^{-1} \bar{\psi}^a \bar{\psi}^c \bar{\lambda}^f \bar{\lambda}^g + \frac{3}{16} \mathcal{F}^{bec} \mathcal{F}^{efg} (\mathcal{I}^{ab})^{-1} \psi^a \psi^c \lambda^f \lambda^g \\
& - \frac{1}{2i} \Big(\frac{1}{4} \mathcal{F}^{abcd} \psi^a \psi^b \lambda^c \lambda^d - \frac{1}{4} \mathcal{F}^{\dagger abcd} \bar{\psi}^a \bar{\psi}^b \bar{\lambda}^c \bar{\lambda}^d \Big) \Big)
\end{aligned}$$

$$+\int d^3x \partial_i (\frac{1}{2}\mathcal{F}^{\dagger ab}\bar{\lambda}^a \bar{\sigma}^i \lambda^b - \frac{i}{2}\mathcal{I}^{ab}\bar{\psi}^a \bar{\sigma}^i \psi^b) \qquad (4.23)$$

where $E^{ai} = v^{a0i}$ and $B^{ai} = \frac{1}{2}\epsilon^{0ijk} v^a_{jk}$ are the SU(2) generalization of the electric and magnetic fields, respectively.

If we call "classical" the terms whose factors are at most second derivatives of $\mathcal{F}$, we see that the first four lines contain only "classical" terms, modulo the two fermions contribution to Π^{ai} (see (4.11)) and the term due to partial integration. The first line are the e.m. terms, and if we write

$$\Pi^{ai} = -(\mathcal{I}^{ab}E^{bi} + \mathcal{R}^{ab}B^{bi}) + \Pi^{ai}_{\mathrm{F}} \qquad (4.24)$$

where Π^{ai}_{F} is the purely quantum two fermions piece, it is easy to see that the "classical" terms combine to give $-\mathcal{I}^{ab}(E^{ai}E^{bi} + B^{ai}B^{bi})$. The second and third lines are the standard terms one would expect for the complex scalars and the fermions, modulo the last term in the third line on which we shall comment in a moment. As promised we reproduced the Yukawa and Higgs potentials, given by the first and second term in the fourth line, respectively. The other terms are purely quantum terms. There we have the two fermions terms coupled to the e.m. fields and momenta and the four fermions term. To check the correctness of these terms we have to consider the correspondent terms in the Lagrangian and compare them with their U(1) limit.

We notice here that, in the last line, we kept a total divergence to explicitly show that we partially integrated the fermionic kinetic terms, in order to fix the phase space $(\bar{\psi}, \bar{\lambda}; \pi_{\bar{\psi}}, \pi_{\bar{\lambda}})$ we started with. It turns out that this total divergence is not symmetric with respect to ψ and λ and this is reflected in the last term of the third line where only λ-terms appear. This means that the Lagrangian we shall obtain by Legendre transforming H will be slightly different from the one expected. Nevertheless the difference will not

affect the conjugate momenta (4.11)-(4.13), therefore the charges (4.19)-(4.21) above constructed are not affected by this asymmetry. Furthermore this problem is entirely due to the non trivial partial integration in the effective case. As explained in the previous Chapter this does not affect the explicit expression of the currents and charges.

A last remark on the computation of this Hamiltonian, is that we extensively exploited the SU(2) generalization of the Poisson brackets defined in the last Chapter. In particular we made use of the "setting independent" formula for the fermions

$$\{\bar{\psi}^a_{\dot{\alpha}}\ ,\ \psi^b_\alpha\}_+ = \{\bar{\lambda}^a_{\dot{\alpha}}\ ,\ \lambda^b_\alpha\}_+ = -i(\mathcal{I}^{ab})^{-1}\sigma^0_{\alpha\dot{\alpha}} \tag{4.25}$$

and the non trivial brackets between bosons and fermions

$$\{\pi^a_A\ ,\ \psi^b_\alpha\}_- \ = \ -\frac{i}{2}(\mathcal{I}^{bc})^{-1}\mathcal{F}^{cad}\psi^d_\alpha \tag{4.26}$$

$$\{\pi^a_{A^\dagger}\ ,\ \psi^b_\alpha\}_- \ = \ +\frac{i}{2}(\mathcal{I}^{bc})^{-1}\mathcal{F}^{\dagger cad}\psi^d_\alpha \tag{4.27}$$

similarly for λ.

We want now to Legendre transform the Hamiltonian (4.23) to obtain the correspondent Lagrangian. Recalling that our metric is $\eta^{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ we have

$$\mathcal{L} = \partial_0 v^{ia}\Pi^{ia} + \partial_0 A^a\pi^a_A + \partial_0 A^{\dagger a}\pi^a_{A^\dagger} + \partial_0\bar{\psi}^a\pi^a_{\bar{\psi}} + \partial_0\bar{\lambda}^a\pi^a_{\bar{\lambda}} - \mathcal{H} \tag{4.28}$$

where $H = \int d^3x\mathcal{H}$ and we discarded the total derivative in the last line of (4.23).

Since for the potential terms (Yukawa, Higgs and four fermions terms) we have simply to reverse the sign, let us concentrate on the other terms.

In the e.m. sector we obtain

$$\mathcal{L}_{\mathrm{e.m.}} \ = \ -\frac{1}{2}\mathcal{I}^{ab}(E^{ia}E^{ib} - B^{ia}B^{ib}) - \mathcal{R}^{ab}E^{ia}B^{ib}$$

$$
\begin{aligned}
&-\frac{1}{2i}\Big(\Pi_{i\mathrm{F}}^{a\psi\lambda}(E^{ia}+iB^{ia})+\Pi_{i\mathrm{F}}^{a\bar\psi\bar\lambda}(E^{ia}-iB^{ia})\Big)\\
&+\frac{3}{2}(\mathcal{I}^{ab})^{-1}\Pi_{\mathrm{F}}^{ia}\Pi_{\mathrm{F}}^{ib}
\end{aligned}
\tag{4.29}
$$

where the second line corresponds to the fifth and sixth lines of the Hamiltonian (4.23), we used the expression (4.24) with $\Pi_{i\mathrm{F}}^{a}\equiv\Pi_{i\mathrm{F}}^{a\psi\lambda}+\Pi_{i\mathrm{F}}^{a\bar\psi\bar\lambda}$ and the last line is the four fermions term that has to be combined with the other four fermions terms.

On the other hand, in the sector with the kinetic terms for the fermions and the scalars we obtain

$$
\begin{aligned}
\mathcal{L}_{\mathrm{scalar-fermi}} \;=\;& -\mathcal{I}^{ab}(\partial_0 A^a\partial^0 A^{\dagger b}+\mathcal{D}_i A^a\mathcal{D}^i A^{\dagger b})\\
&-i\mathcal{I}^{ab}(\psi^a\sigma^0\partial_0\bar\psi^b+\psi^a\sigma^i\mathcal{D}_i\bar\psi^b+\lambda^a\sigma^0\partial_0\bar\lambda^b+\lambda^a\sigma^i\mathcal{D}_i\bar\lambda^b)\\
&-\frac{1}{2}(\partial_\mu\mathcal{F}^{\dagger ab})\bar\lambda^a\bar\sigma^\mu\lambda^b-\frac{1}{2}(\partial_0\mathcal{F}^{\dagger ab})\bar\psi^a\bar\sigma^0\psi^b
\end{aligned}
$$

where we used the $\pi_{A^\dagger}\neq(\pi_A)^\dagger$ given in (4.13). Note again here that ψ and λ in the last line, do not have the same factor, as explained above.

If we reintroduce v^0 in these expressions, our Lagrangian density is given by

$$
\mathcal{L}\equiv\mathcal{L}_1+\mathcal{L}_2 \tag{4.30}
$$

where

$$
\begin{aligned}
\mathcal{L}_1 \;=\;& \frac{1}{2i}\Big[-\frac{1}{4}\mathcal{F}^{ab}v^{a\mu\nu}\hat v^{b}_{\mu\nu}+\frac{1}{4}\mathcal{F}^{\dagger ab}v^{a\mu\nu}\hat v^{\dagger b}_{\mu\nu}\Big]\\
&-\mathcal{I}^{ab}\Big[\mathcal{D}_\mu A^a\mathcal{D}^\mu A^{\dagger b}+i\psi^a\,\not{\mathcal{D}}\bar\psi^b+i\lambda^a\,\not{\mathcal{D}}\bar\lambda^b\\
&-\frac{1}{\sqrt2}\epsilon^{adc}(A^c\bar\psi^b\bar\lambda^d+A^{\dagger c}\psi^b\lambda^d)+\frac{1}{2}\epsilon^{acd}\epsilon^{bfg}A^cA^{\dagger d}A^fA^{\dagger g}\Big]\\
&-\frac{1}{2}(\partial_\mu\mathcal{F}^{\dagger ab})\bar\lambda^a\bar\sigma^\mu\lambda^b-\frac{1}{2}(\partial_0\mathcal{F}^{\dagger ab})\bar\psi^a\bar\sigma^0\psi^b
\end{aligned}
\tag{4.31}
$$

contains the "classical" terms, and

$$
\mathcal{L}_2 \;=\; \frac{1}{2i}\Big[\frac{1}{\sqrt2}\mathcal{F}^{abc}\lambda^a\sigma^{\mu\nu}\psi^b v^c_{\mu\nu}-\frac{1}{\sqrt2}\mathcal{F}^{\dagger abc}\bar\lambda^a\bar\sigma^{\mu\nu}\bar\psi^b v^c_{\mu\nu}\Big]
$$

$$
\begin{aligned}
&+\frac{3}{16}(\mathcal{I}^{ab})^{-1}\Big[\mathcal{F}^{acd}\mathcal{F}^{bef}(\psi^d\psi^f\lambda^c\lambda^e-\psi^d\lambda^e\lambda^c\psi^f)-\mathcal{F}^{gbc}\mathcal{F}^{gef}\psi^a\psi^c\lambda^e\lambda^f\\
&+\mathcal{F}^{\dagger acd}\mathcal{F}^{\dagger bef}(\bar\psi^d\bar\psi^f\bar\lambda^c\bar\lambda^e-\bar\psi^d\bar\lambda^e\bar\lambda^c\bar\psi^f)-\mathcal{F}^{\dagger gbc}\mathcal{F}^{\dagger gef}\bar\psi^a\bar\psi^c\bar\lambda^e\bar\lambda^f\\
&+\mathcal{F}^{\dagger acd}\mathcal{F}^{\dagger bef}(\lambda^c\sigma^0\bar\psi^f\psi^d\sigma^0\bar\lambda^e-\lambda^c\sigma^0\bar\lambda^e\psi^d\sigma^0\bar\psi^f)\Big]\\
&-\frac{1}{16}(\mathcal{I}^{ab})^{-1}\mathcal{F}^{\dagger acd}\mathcal{F}^{bef}(\bar\psi^c\bar\psi^d\psi^e\psi^f+\bar\lambda^c\bar\lambda^d\lambda^e\lambda^f)\\
&+\frac{1}{2i}(\frac{1}{4}\mathcal{F}^{abcd}\psi^a\psi^b\lambda^c\lambda^d-\frac{1}{4}\mathcal{F}^{\dagger abcd}\bar\psi^a\bar\psi^b\bar\lambda^c\bar\lambda^d) \qquad (4.32)
\end{aligned}
$$

contains the purely quantum terms. The second and third lines of (4.32) come from the combination of $\frac{3}{2}\Pi_{\rm F}$ and the four fermions in the Hamiltonian, whereas the fourth line comes from $\frac{3}{2}\Pi_{\rm F}$ alone. In Appendix F we shall show that the factors are in agreement with the U(1) correspondent ones.

Note also that the above given expression for the SW SU(2) high-energy effective Lagrangian is in agreement with the one obtained directly by superfield expansion in [19].

What is left is to produce the Gauss law descending from this Lagrangian. At this end we have only to consider the terms in the Lagrangian that contain the Lagrange multiplier v^{g0} and define the associated Gauss constraint as we did in the U(1) sector (see (3.9)). We obtain

$$
\begin{aligned}
0=\frac{\partial\mathcal{L}}{\partial v^{g0}} &= \frac{1}{2i}[\partial_i(-\mathcal{F}^{gb}\hat v_0^{bi}+\sqrt{2}\mathcal{F}^{gbc}\lambda^b\sigma_0^i\psi^c)\\
&\quad -\epsilon^{gad}v^{id}\mathcal{F}^{ab}\hat v_{i0}^b+\epsilon^{gcd}\sqrt{2}\mathcal{F}^{abc}\lambda^a\sigma_0^i\psi^b v_i^d-h.c.]\\
&+ \epsilon^{gac}\mathcal{I}^{ab}(A^c\mathcal{D}_0A^{\dagger b}+A^{\dagger c}\mathcal{D}_0A^b+i\psi^b\sigma_0\bar\psi^c+i\lambda^b\sigma_0\bar\lambda^c) \qquad (4.33)
\end{aligned}
$$

Recalling the definition of the conjugate momentum Π^{gi} of v_i^g given in (4.11) and the definition of the covariant derivative, $\mathcal{D}_\mu X^a=\partial_\mu X^a+\epsilon^{abc}v_\mu^b X^c$, we see that the first two lines give $\mathcal{D}_i\Pi^{gi}$. Thus we have

$$
\mathcal{D}_i\Pi^{ig}=-\epsilon^{gac}\mathcal{I}^{ab}(A^c\mathcal{D}^0A^{\dagger b}+A^{\dagger c}\mathcal{D}^0A^b+i\psi^b\sigma^0\bar\psi^c+i\lambda^b\sigma^0\bar\lambda^c) \qquad (4.34)
$$

which is the required Gauss law.

4.3 Computation of the central charge

We can now compute the central charge for the SU(2) theory. As we did in the U(1) sector, we first compute the Poisson brackets of $\epsilon_1 Q_1$ and $\epsilon_2 Q_2$ given in (4.19) and (4.20), respectively. The non zero contributions are given by

$$
\begin{aligned}
\{\epsilon_1 Q_1, \epsilon_2 Q_2\}_- &= \int d^3x d^3y \Big(\{\Pi^{ai}, v^{*0jd}\}_- \delta_1 v_i^a \frac{i}{2} \mathcal{F}^{\dagger cd} \epsilon_2 \sigma_j \bar{\psi}^c \\
&+ \{v^{*0ib}, \Pi^{cj}\}_- \delta_2 v_j^c \frac{i}{2} \mathcal{F}^{\dagger ab} \epsilon_1 \sigma_i \bar{\lambda}^a && (4.35) \\
&+ \delta_1 \bar{\psi}^a_{\dot{\alpha}} \{\pi^{\dot{\alpha} a}_{\bar{\psi}}, \bar{\psi}^e \bar{\psi}^f\}_- \frac{1}{2\sqrt{2}} \mathcal{F}^{\dagger def} \epsilon_2 \sigma^0 \bar{\lambda}^d \\
&+ \delta_2 \bar{\lambda}^d_{\dot{\alpha}} \{\pi^{\dot{\alpha} d}_{\bar{\lambda}}, \bar{\lambda}^b \bar{\lambda}^c\}_- \frac{1}{2\sqrt{2}} \mathcal{F}^{\dagger abc} \epsilon_1 \sigma^0 \bar{\psi}^a && (4.36) \\
&+ \Pi^{ai} \delta_2 \bar{\lambda}^b_{\dot{\alpha}} \{\delta_1 v_i^a, \pi^{\dot{\alpha} b}_{\bar{\lambda}}\}_- \\
&+ \Pi^{bj} \delta_1 \bar{\psi}^b_{\dot{\alpha}} \{\pi^{\dot{\alpha} a}_{\bar{\psi}}, \delta_2 v_j^b\}_- && (4.37) \\
&+ \delta_1 \bar{\psi}^a_{\dot{\alpha}} \{\pi^{\dot{\alpha} a}_{\bar{\psi}}, \epsilon_2 \sigma_j \bar{\psi}^c\}_- \frac{i}{2} \mathcal{F}^{\dagger cd} v^{*0jd} \\
&+ \delta_2 \bar{\lambda}^b_{\dot{\alpha}} \{\epsilon_1 \sigma_i \bar{\lambda}^a, \pi^{\dot{\alpha} b}_{\bar{\lambda}}\}_- \frac{i}{2} \mathcal{F}^{\dagger ab} v^{*0ib} && (4.38) \\
&+ \{\Pi^{ai}, \delta_2 \bar{\lambda}^b_{\dot{\alpha}}\}_- \delta_1 v_i^a \pi^{\dot{\alpha} b}_{\bar{\lambda}} \\
&+ \{\delta_1 \bar{\psi}^a_{\dot{\alpha}}, \Pi^{bj}\}_- \delta_2 v_j^b \pi^{\dot{\alpha} a}_{\bar{\psi}} && (4.39) \\
&+ i\mathcal{I}^{ef} \delta_1 \bar{\psi}^a_{\dot{\alpha}} \{\pi^{\dot{\alpha} a}_{\bar{\psi}}, \epsilon_2 \sigma^0 \bar{\psi}^f\}_- \epsilon^{egh} A^g A^{h\dagger} \\
&+ i\mathcal{I}^{ab} \delta_2 \bar{\lambda}^c_{\dot{\alpha}} \{\epsilon_1 \sigma^0 \bar{\lambda}^b, \pi^{\dot{\alpha} c}_{\bar{\lambda}}\}_- \epsilon^{ade} A^d A^{e\dagger} && (4.40) \\
&+ \{\pi_A^b, \mathcal{I}^{ef}\} \delta_1 A^b i\epsilon_2 \sigma^0 \bar{\psi}^f \epsilon^{egh} A^g A^{\dagger h} \\
&+ \{\mathcal{I}^{ab}, \pi_A^e\} \delta_2 A^e i\epsilon_1 \sigma^0 \bar{\lambda}^b \epsilon^{acd} A^c A^{\dagger d} && (4.41) \\
&+ \{\pi_A^b, A^g\} \delta_1 A^b i\mathcal{I}^{ef} \epsilon_2 \sigma^0 \bar{\psi}^f \epsilon^{egh} A^{\dagger h} \\
&+ \{A^c, \pi_A^e\} \delta_2 A^e i\mathcal{I}^{ab} \epsilon_1 \sigma^0 \bar{\lambda}^b \epsilon^{acd} A^{\dagger d} \Big) && (4.42)
\end{aligned}
$$

The eight terms (four pairs) (4.35) - (4.38) are simply the SU(2) *version* of the Abelian computation. On the one hand, terms (4.35) and (4.36) give

$$\int d^3x \partial_i [i\mathcal{F}^{ab\dagger}(\epsilon_1\sigma^0\bar{\psi}^a\epsilon_2\sigma^i\bar{\lambda}^b - \epsilon_2\sigma^0\bar{\lambda}^a\epsilon_1\sigma^i\bar{\psi}^b)] \tag{4.43}$$

where the terms (4.35) give $\mathcal{F}^{\dagger ab}\partial(\bar{\psi}^a\bar{\lambda}^b)$ type of term and the terms (4.36) give $(\partial\mathcal{F}^{\dagger ab})\bar{\psi}^a\bar{\lambda}^b$ type of term. Note that there is no SU(2) contribution coming from the covariant derivative $\mathcal{D}A^\dagger$.

On the other hand, terms (4.37) and (4.38) give

$$\int d^3x \Big(\partial_i[2\sqrt{2}\Pi^{ai}A^{\dagger a}+\sqrt{2}v^{*0ia}A_D^{\dagger a}]+2\sqrt{2}(\mathcal{D}_i\Pi^{ai})A^{a\dagger}+\sqrt{2}(\mathcal{D}_i v^{*0ia})A_D^{\dagger a}\Big)\epsilon_1\epsilon_2 \tag{4.44}$$

where again $B^{ai} = \frac{1}{2}\epsilon^{0ijk}v^a_{jk}$ and we introduced the SW dual of $A^{a\dagger}$, $A_D^{a\dagger} \equiv \mathcal{F}^{a\dagger}$. The Bianchi identities $\mathcal{D}_i v^{*0ia} = 0$ can be applied in this case as well, thus we expect the eight new terms (four pairs) (4.39)-(4.42) to contribute to the Gauss law only. Let us look at them one by one.

The terms (4.39) give

$$\int d^3x \; i\sqrt{2}\mathcal{I}^{ae}\epsilon^{acd}A^{\dagger d}(\epsilon_1\sigma^i\bar{\sigma}^0\psi^e\epsilon_2\sigma_i\bar{\psi}^c + \epsilon_2\sigma^i\bar{\sigma}^0\lambda^e\epsilon_1\sigma_i\bar{\lambda}^c) \tag{4.45}$$

the terms (4.40) give

$$\begin{aligned} &-\int d^3x \; \sqrt{2}\mathcal{I}^{ab}(\mathcal{D}_\mu A^{\dagger b})\epsilon^{ade}A^d A^{\dagger e}(\epsilon_1\sigma^0\bar{\sigma}^\mu\epsilon_2 + \epsilon_1\sigma^\mu\bar{\sigma}^0\epsilon_2) \\ = \;\; &-\int d^3x 2\sqrt{2}\pi^a_A\epsilon^{ade}A^d A^{\dagger e}\epsilon_1\epsilon_2 \end{aligned} \tag{4.46}$$

where we used $\pi^a_A = -\mathcal{I}^{ab}\partial^0 A^{\dagger b}$ and $(\sigma^0\bar{\sigma}^\mu + \sigma^\mu\bar{\sigma}^0)_\alpha^\beta = -2\eta^{0\mu}\delta_\alpha^\beta$.

The terms (4.41) give

$$\int d^3x \; -\frac{1}{\sqrt{2}}\mathcal{F}^{efb}\epsilon^{egh}A^g A^{\dagger h}(\epsilon_1\psi^b\epsilon_2\sigma^0\bar{\psi}^f + \epsilon_2\lambda^b\epsilon_1\sigma^0\bar{\lambda}^f) \tag{4.47}$$

Finally terms (4.42) give

$$\int d^3x \; - i\sqrt{2}\mathcal{I}^{ef}\epsilon^{egh}A^{\dagger h}(\epsilon_1\psi^g\epsilon_2\sigma^0\bar{\psi}^f + \epsilon_2\lambda^g\epsilon_1\sigma^0\bar{\lambda}^f) \tag{4.48}$$

As usual we now get rid of ϵ_1 and ϵ_2, sum over the spinor indices ($\{\epsilon_1 Q_1, \epsilon_2 Q_2\}_- = -\epsilon_1^\alpha \epsilon_2^\beta \{Q_{1\alpha}, Q_{2\beta}\}_+$ and $\epsilon_{\alpha\beta}\epsilon^{\alpha\beta} = -2$) and write the centre as $Z = \frac{i}{4}\epsilon^{\alpha\beta}\{Q_{1\alpha}, Q_{2\beta}\}_+$. Collecting all the contributions we obtain

$$
\begin{aligned}
Z &= \int d^3x \Big(\partial_i [i\sqrt{2}(\Pi^{ai} A^{\dagger a} + B^{ai} A_D^{\dagger a}) - \mathcal{F}^{\dagger ab} \bar{\psi}^a \bar{\sigma}^{i0} \bar{\lambda}^b] \\
&+ i\sqrt{2}(\mathcal{D}_i \Pi^{ai}) A^{\dagger a} \\
&- \sqrt{2}\mathcal{I}^{be}\epsilon^{bcd} A^{\dagger d}(\psi^e \sigma^0 \bar{\psi}^c + \lambda^e \sigma^0 \bar{\lambda}^c) \\
&- \frac{1}{2\sqrt{2}}(\mathcal{I}^{be}\epsilon^{bcd} + \mathcal{I}^{bc}\epsilon^{bed}) A^{\dagger d}(\psi^e \sigma^0 \bar{\psi}^c + \lambda^e \sigma^0 \bar{\lambda}^c) \\
&- i\sqrt{2}\epsilon^{abc} A^b A^{\dagger c} \pi_A^a \\
&+ \frac{i}{4\sqrt{2}} \mathcal{F}^{abc}\epsilon^{ade} A^d A^{\dagger e}(\psi^c \sigma^0 \bar{\psi}^b + \lambda^c \sigma^0 \bar{\lambda}^b) \Big)
\end{aligned}
\tag{4.49}
$$

As shown in Appendix F we can recast the terms in the last line into $\mathcal{F}^{ab}$ type of terms and combine them with the similar terms (remember that $2i\mathcal{I}^{ab} = (\mathcal{F}^{ab} - \mathcal{F}^{\dagger ab})$). The result of this recombination is given by

$$
\begin{aligned}
Z &= \int d^3x \Big(\partial_i [i\sqrt{2}(\Pi^{ai} A^{\dagger a} + B^{ai} A_D^{\dagger a}) - \mathcal{F}^{\dagger ab} \bar{\psi}^a \bar{\sigma}^{i0} \bar{\lambda}^b] \\
&+ i\sqrt{2}[(D_i \Pi^{ai}) A^{\dagger a} + i\mathcal{I}^{be}\epsilon^{bcd} A^{\dagger d}(\psi^e \sigma^0 \bar{\psi}^c + \lambda^e \sigma^0 \bar{\lambda}^c) \\
&\quad -\epsilon^{abc} A^b A^{\dagger c} \pi_A^a] \Big)
\end{aligned}
\tag{4.50}
$$

We see from here that the terms which are not a total divergence, given in the second and third lines above, simply cancel due to the Gauss law (4.34) obtained in the previous Section

$$
D_i \Pi^{ai} = -\epsilon^{abc}\mathcal{I}^{bd}(A^c \partial^0 A^{\dagger d} + A^{\dagger c}\partial^0 A^d + i(\psi^d \sigma^0 \bar{\psi}^c + \lambda^d \sigma^0 \bar{\lambda}^c)) \tag{4.51}
$$

Eventually we are left with the surface terms that vanish when the SU(2) gauge symmetry is not broken down to U(1). If we break the symmetry along a flat direction of the Higgs potential, say $a = 3$, we recover the same result we found in the U(1) sector. In other words we see that on the sphere

at infinity

$$
\begin{aligned}
Z &= \int d^2\vec{\Sigma}\cdot[i\sqrt{2}(\vec{\Pi}^a A^{\dagger a}+\frac{1}{4\pi}\vec{B}^a A_D^{\dagger a})-\frac{1}{4\pi}\mathcal{F}^{\dagger ab}\bar{\psi}^a\vec{\bar{\sigma}}\bar{\lambda}^b)] \\
&\to i\sqrt{2}\int d^2\vec{\Sigma}\cdot(\vec{\Pi}^3 A^{\dagger 3}+\frac{1}{4\pi}\vec{B}^3 A_D^{\dagger 3}) \qquad (4.52)
\end{aligned}
$$

where $\vec{\bar{\sigma}} \equiv (\bar{\sigma}^{01}, \bar{\sigma}^{02}, \bar{\sigma}^{03})$ and we reintroduced the factor 4π. We made the usual assumption that the bosonic massive fields in the SU(2)/U(1) sector ($a = 1, 2$) and all the fermionic fields fall off faster than r^3, whereas the scalar massless field ($a = 3$) and its dual tend to their Higgs v.e.v.'s a^* and a_D^*, respectively.

We conclude that the fields in the massive sector, have no effect on the mass formula.

Appendix A

Proof of Noether Theorem

The following proof is based on Ref.s [6] and [45].

Let us consider the Action

$$\mathcal{A}_\Omega = \int_\Omega d^4x \mathcal{L}(\Phi_i, \partial\Phi_i) \tag{A.1}$$

where Ω is the space-time volume of integration. The infinitesimal transformations of the coordinates, of the fields and of the derivatives of the fields are given respectively by

$$\begin{aligned} x_\mu &\rightarrow x'_\mu = x_\mu + \delta x_\mu & \text{(A.2)} \\ \Phi_i(x) &\rightarrow \Phi'_i(x') = \Phi_i(x) + \delta\Phi_i(x) & \text{(A.3)} \\ \partial_\mu \Phi_i(x) &\rightarrow \partial'_\mu \Phi'_i(x') = \partial_\mu \Phi_i(x) + \delta\partial_\mu \Phi_i(x) & \text{(A.4)} \end{aligned}$$

note that δ does not commute with the derivatives.

When we act with this transformation the Action changes to

$$\mathcal{A}'_{\Omega'} = \int_{\Omega'} d^4x' \mathcal{L}(\Phi'_i, \partial'\Phi'_i) \tag{A.5}$$

If the transformation is a symmetry we have $\mathcal{A}'_{\Omega'} - \mathcal{A}_\Omega = 0$, therefore at the

first order we obtain

$$
\begin{aligned}
0 &= \mathcal{A}'_{\Omega'} - \mathcal{A}_{\Omega} \\
&= \int_{\Omega} d^4x \Big[(1+\partial_\rho \delta x^\rho)\Big(\mathcal{L}(\Phi_i, \partial\Phi_i) + \frac{\partial \mathcal{L}}{\partial \Phi_i}\delta\Phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)}\delta\partial_\mu\Phi_i\Big) - \mathcal{L}\Big] \\
&= \int_{\Omega} d^4x \Big(\frac{\partial \mathcal{L}}{\partial \Phi_i}\delta\Phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)}\delta\partial_\mu\Phi_i + \mathcal{L}\partial_\rho x^\rho\Big) \qquad \text{(A.6)}
\end{aligned}
$$

where $(1+\partial_\rho \delta x^\rho)$ is the Jacobian of the change of coordinates from x' to x at the first order.

Let us now introduce another variation δ^* that commutes with the derivatives. If we do so we can write

$$
\delta\Phi_i = \partial_\mu \Phi_i(x)\delta x^\mu + \delta^*\Phi_i \quad \text{and} \quad \delta(\partial_\mu\Phi_i) = \partial_\mu\partial_\nu\Phi_i(x)\delta x^\nu + \delta^*\partial_\mu\Phi_i \quad \text{(A.7)}
$$

Substituting these back in (A.6) we obtain

$$
\begin{aligned}
&\int_{\Omega} d^4x \Big[\partial_\mu\Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)}\delta^*\Phi_i\Big) + \Big(\frac{\partial \mathcal{L}}{\partial \Phi_i}\partial_\mu\Phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi_i)}\partial_\mu\partial_\nu\Phi_i\Big)\delta x^\mu + \mathcal{L}\partial_\mu\delta x^\mu\Big] \\
= &\int_{\Omega} d^4x \Big[\partial_\mu\Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)}\delta^*\Phi_i\Big) + \Big(\frac{\partial \mathcal{L}}{\partial x^\mu}\delta x^\mu + \mathcal{L}\partial_\mu\delta x^\mu\Big)\Big] \qquad \text{(A.8)} \\
= &\int_{\Omega} d^4x (\text{E.L.})_i \delta^*\Phi^i
\end{aligned}
$$

which finally gives the wanted conservation law on-shell $\partial_\mu J^\mu = 0$ where

$$
J^\mu = \Pi^\mu_i \delta^*\Phi^i + \mathcal{L}\delta x^\mu \qquad \text{(A.9)}
$$

This leads to the identification $V^\mu = -\mathcal{L}\delta x^\mu$ introduced in Section 1.1. If we write back the space-time dependent variations $\delta\Phi^i$ we obtain

$$
\begin{aligned}
J^\mu &= \Pi^\mu_i \delta\Phi^i - (\Pi^\mu_i \partial^\nu\Phi_i - \eta^{\mu\nu}\mathcal{L})\delta x_\nu \\
&= \Pi^\mu_i \delta\Phi^i - T^{\mu\nu}\delta x_\nu \qquad \text{(A.10)}
\end{aligned}
$$

that leads to the definition (1.8) of the energy-momentum tensor $T^{\mu\nu}$.

Appendix B

Notation and Spinor Algebra

Let us say here that in Susy conventions and notations are not a trivial matter at all. We follow the conventions of Wess and Bagger [9] with no changes. Fortunately these conventions are becoming more and more popular and this is one of the reasons why we chose them. Rather than filling pages with well known formulae we refer to the Appendices A and B in [9]. Here we shall comment on some of those conventions and show the formulae more relevant for our computations.

B.1 Crucial conventions

The spinors are Weyl two components in Van der Waerden notation. Spinors with undotted indices transform under the representation $(\frac{1}{2}, 0)$ of $SL(2, \mathbf{C})$ and spinors with dotted indices transform under the conjugate representation $(0, \frac{1}{2})$. The relations between Dirac, Majorana and Weyl spinors are given by

$$\Psi_{\text{Dirac}} = \begin{pmatrix} \psi_\alpha \\ \bar{\lambda}^{\dot{\alpha}} \end{pmatrix} \quad \Psi_{\text{Majorana}} = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} \tag{B.1}$$

The sigma matrices are standard Pauli matrices

$$\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{B.2}$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{B.3}$$

The relation with the gamma matrices is given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{B.4}$$

The metric is $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. To rise and lower the spinor indices we use $\epsilon_{\alpha\beta}$ and $\epsilon^{\alpha\beta}$, where $\epsilon_{21} = \epsilon^{12} = -\epsilon_{12} = -\epsilon^{21} = 1$. Also $\epsilon_{0123} = -1$. The position of the spinor indices is not negotiable and is given once and for all by

$$\sigma^\mu_{\alpha\dot{\alpha}} \quad \bar{\sigma}^{\mu\,\dot{\alpha}\alpha} \quad \sigma^{\mu\nu\,\beta}_{\alpha} \quad \bar{\sigma}^{\mu\nu\,\dot{\alpha}}_{\dot{\beta}} \tag{B.5}$$

where

$$\begin{aligned} \sigma^{\mu\nu\,\beta}_{\alpha} &= \frac{1}{4}(\sigma^\mu_{\alpha\dot{\alpha}}\bar{\sigma}^{\dot{\alpha}\beta\,\nu} - \sigma^\nu_{\alpha\dot{\alpha}}\bar{\sigma}^{\dot{\alpha}\beta\,\mu}) && \text{(B.6)} \\ \bar{\sigma}^{\mu\nu\,\dot{\alpha}}_{\dot{\beta}} &= \frac{1}{4}(\bar{\sigma}^{\dot{\alpha}\alpha\,\mu}\sigma^\nu_{\alpha\dot{\beta}} - \bar{\sigma}^{\dot{\alpha}\alpha\,\nu}\sigma^\mu_{\alpha\dot{\beta}}) && \text{(B.7)} \end{aligned}$$

From σ to $\bar{\sigma}$ and *vice versa*:

$$\begin{aligned} &\sigma^\mu_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{\mu\,\dot{\beta}\beta} \quad \bar{\sigma}^{\mu\,\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot{\beta}} && \text{(B.8)} \\ &\epsilon_{\alpha\beta}\bar{\sigma}^{\dot{\alpha}\beta}_\mu = \epsilon^{\dot{\alpha}\dot{\beta}}\sigma_{\mu\alpha\dot{\beta}} \quad \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{\dot{\alpha}\beta}_\mu = \epsilon^{\alpha\beta}\sigma_{\mu\alpha\dot{\beta}} && \text{(B.9)} \end{aligned}$$

To raise and lower spinor indices use A(9) in [9] always matching the indices from left to right as follows:

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta \quad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta \tag{B.10}$$

of course

$$\psi^\beta \epsilon_{\beta\alpha} = \epsilon_{\beta\alpha}\psi^\beta = -\epsilon_{\alpha\beta}\psi^\beta = -\psi_\alpha \tag{B.11}$$

As explained in Section B.4 momenta are on a different footing and the convention to raise and lower their indices is the opposite to the standard one. Namely

$$\pi^\alpha = \epsilon^{\beta\alpha}\pi_\beta \quad \pi_\alpha = \epsilon_{\beta\alpha}\pi^\beta \tag{B.12}$$

Quantities with one spinor index are grassmanian variables thus anti-commute:

$$\psi_\alpha\chi_\beta = -\chi_\beta\psi_\alpha, \quad \bar\psi_{\dot\alpha}\bar\chi_{\dot\beta} = -\bar\chi_{\dot\beta}\bar\psi_{\dot\alpha}, \quad \bar\psi_{\dot\alpha}\chi_\beta = -\chi_\beta\bar\psi_{\dot\alpha} \tag{B.13}$$

But some care is needed due to the (subtle) convention

$$\psi\chi \equiv \psi^\alpha\chi_\alpha = -\psi_\alpha\chi^\alpha \tag{B.14}$$

and

$$\bar\psi\bar\chi \equiv \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} = -\bar\psi^{\dot\alpha}\bar\chi_{\dot\alpha} \tag{B.15}$$

that leads to

$$(\psi\chi)^\dagger = \bar\psi\bar\chi \tag{B.16}$$

with no minus sign. Note that $\psi\chi = \chi\psi$ ($\bar\psi\bar\chi = \bar\chi\bar\psi$) but $\pi\chi = -\chi\pi$ ($\bar\pi\bar\chi = -\bar\chi\bar\pi$) where π is a momentum. Explicitly writing the indices that means: $\pi^\alpha\chi_\alpha = \pi_\alpha\chi^\alpha$ and $\bar\pi_{\dot\alpha}\bar\chi^{\dot\alpha} = \bar\pi^{\dot\alpha}\bar\chi_{\dot\alpha}$.
Quantities with two spinor indices are c-number matrices

$$\epsilon_{\alpha\beta}, \quad \epsilon_{\dot\alpha\dot\beta}, \quad \sigma^\mu_{\alpha\dot\beta}, \quad \bar\sigma^{\mu\dot\alpha\beta}, \quad (\sigma^{\mu\nu})_\alpha{}^\beta, \quad (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}, \tag{B.17}$$

For instance the (anti)commutator of σ^μ and $\bar\sigma^\nu$ is with respect to the Minkowski indices μ, ν.
Other formulae:

$$\partial\!\!\!/_{\alpha\dot\alpha}\bar\partial\!\!\!/^{\dot\alpha\beta} = -\delta^\beta_\alpha\Box \quad \bar\partial\!\!\!/^{\dot\alpha\alpha}\,\partial\!\!\!/_{\alpha\dot\beta} = -\delta^{\dot\alpha}_{\dot\beta}\Box \tag{B.18}$$

where $\not{\partial}_{\alpha\dot{\alpha}} \equiv \sigma^{\mu}_{\alpha\dot{\alpha}}\partial_\mu$ and $\bar{\not{\partial}}^{\dot{\alpha}\alpha} \equiv \bar{\sigma}^{\dot{\alpha}\alpha}_{\mu}\partial^\mu$.

Also $\psi\sigma^{\mu\nu}\chi = -\chi\sigma^{\mu\nu}\psi$, $(\psi\sigma^{\mu\nu}\chi)^\dagger = -\bar{\chi}\bar{\sigma}^{\mu\nu}\bar{\psi}$, $(\psi\ \not{\partial}\bar{\psi})^\dagger = -\bar{\psi}\bar{\not{\partial}}\psi$.

B.2 Useful algebra

Beside the Fierz identities given in (B.13) in [9] we also find

$$\psi^\alpha\lambda^\beta - \psi^\beta\lambda^\alpha = -\epsilon^{\alpha\beta}\psi\lambda \quad \psi_\alpha\lambda_\beta - \psi_\beta\lambda_\alpha = \epsilon_{\alpha\beta}\psi\lambda \tag{B.19}$$

$$\psi_\alpha\lambda^\beta - \psi^\beta\lambda_\alpha = -\delta^\beta_\alpha\psi\lambda \quad \psi^\alpha\lambda_\beta - \psi_\beta\lambda^\alpha = \delta^\alpha_\beta\psi\lambda \tag{B.20}$$

$$\bar{\psi}^{\dot{\alpha}}\bar{\lambda}^{\dot{\beta}} - \bar{\psi}^{\dot{\beta}}\bar{\lambda}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}\bar{\lambda} \quad \bar{\psi}_{\dot{\alpha}}\bar{\lambda}_{\dot{\beta}} - \bar{\psi}_{\dot{\beta}}\bar{\lambda}_{\dot{\alpha}} = -\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}\bar{\lambda} \tag{B.21}$$

$$\bar{\psi}_{\dot{\alpha}}\bar{\lambda}^{\dot{\beta}} - \bar{\psi}^{\dot{\beta}}\bar{\lambda}_{\dot{\alpha}} = \delta^{\dot{\beta}}_{\dot{\alpha}}\bar{\psi}\bar{\lambda} \quad \bar{\psi}^{\dot{\alpha}}\bar{\lambda}_{\dot{\beta}} - \bar{\psi}_{\dot{\beta}}\bar{\lambda}^{\dot{\alpha}} = -\delta^{\dot{\alpha}}_{\dot{\beta}}\bar{\psi}\bar{\lambda} \tag{B.22}$$

Using the definitions and the properties given in (A.11), (A.14) and (A.15) in [9] we find :

$$\sigma^{\mu\nu}\sigma^\lambda = \frac{1}{2}(-\eta^{\lambda\nu}\sigma^\mu + \eta^{\lambda\mu}\sigma^\nu + i\epsilon^{\lambda\mu\nu\kappa}\sigma_\kappa) \tag{B.23}$$

$$\sigma^\mu\bar{\sigma}^{\nu\lambda} = \frac{1}{2}(\eta^{\mu\lambda}\sigma^\nu - \eta^{\mu\nu}\sigma^\lambda + i\epsilon^{\mu\nu\lambda\kappa}\sigma_\kappa) \tag{B.24}$$

$$\bar{\sigma}^{\mu\nu}\bar{\sigma}^\lambda = \frac{1}{2}(-\eta^{\lambda\nu}\bar{\sigma}^\mu + \eta^{\lambda\mu}\bar{\sigma}^\nu - i\epsilon^{\lambda\mu\nu\kappa}\bar{\sigma}_\kappa) \tag{B.25}$$

$$\bar{\sigma}^\mu\sigma^{\nu\lambda} = \frac{1}{2}(\eta^{\mu\lambda}\bar{\sigma}^\nu - \eta^{\mu\nu}\bar{\sigma}^\lambda - i\epsilon^{\mu\nu\lambda\kappa}\bar{\sigma}_\kappa) \tag{B.26}$$

which imply

$$\sigma^{\mu\nu}\sigma_\nu = \sigma_\nu\bar{\sigma}^{\nu\mu} = -\frac{3}{2}\sigma^\mu \qquad \bar{\sigma}^{\mu\nu}\bar{\sigma}_\nu = \bar{\sigma}_\nu\sigma^{\nu\mu} = -\frac{3}{2}\bar{\sigma}^\mu \tag{B.27}$$

Very useful is the version of the previous identities with free spinor indices

$$\sigma^{\mu\nu}{}_{\alpha}{}^{\beta}\sigma_{\nu\,\gamma\dot{\gamma}} = \frac{1}{2}(\sigma^\mu_{\delta\dot{\gamma}}\epsilon_{\gamma\alpha}\epsilon^{\beta\delta} - \sigma^\mu_{\alpha\dot{\gamma}}\delta^\beta_\gamma) \tag{B.28}$$

$$\sigma^{\mu\nu}{}_{\alpha}{}^{\beta}\bar{\sigma}^{\dot{\alpha}\gamma}_{\nu} = \frac{1}{2}(\bar{\sigma}^{\mu\,\dot{\alpha}\delta}\epsilon_{\alpha\delta}\epsilon^{\beta\gamma} + \bar{\sigma}^{\mu\,\dot{\alpha}\beta}\delta^\gamma_\alpha) \tag{B.29}$$

$$\bar{\sigma}^{\mu\nu\,\dot{\alpha}}{}_{\dot{\beta}}\bar{\sigma}^{\dot{\gamma}\gamma}_{\nu} = \frac{1}{2}(\bar{\sigma}^{\mu\,\dot{\delta}\gamma}\epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon_{\dot{\delta}\dot{\beta}} - \bar{\sigma}^{\mu\,\dot{\alpha}\gamma}\delta^{\dot{\gamma}}_{\dot{\beta}}) \tag{B.30}$$

$$\bar{\sigma}^{\mu\nu\,\dot{\alpha}}{}_{\dot{\beta}}\sigma_{\nu\,\alpha\dot{\gamma}} = \frac{1}{2}(\sigma^\mu{}_{\alpha\dot{\delta}}\epsilon^{\dot{\alpha}\dot{\delta}}\epsilon_{\dot{\beta}\dot{\gamma}} + \sigma^\mu{}_{\alpha\dot{\beta}}\delta^{\dot{\alpha}}_{\dot{\gamma}}) \tag{B.31}$$

We also have

$$\begin{aligned}
\sigma^{\mu\nu\beta}_{\alpha}\bar{\sigma}^{\dot{\alpha}}_{\mu\nu\dot{\beta}} &= -\delta^{\dot{\alpha}}_{\dot{\beta}}\delta^{\beta}_{\alpha} &\quad (B.32)\\
\sigma^{0\mu\beta}_{\alpha}\sigma^{\delta}_{\gamma 0\mu} &= -\frac{1}{4}(\epsilon_{\alpha\gamma}\epsilon^{\beta\delta} + \delta^{\delta}_{\alpha}\delta^{\beta}_{\gamma}) &\quad (B.33)\\
\bar{\sigma}^{0\mu\dot{\alpha}}_{\dot{\beta}}\bar{\sigma}^{\dot{\gamma}}_{0\mu\dot{\delta}} &= -\frac{1}{4}(\epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon_{\dot{\beta}\dot{\delta}} + \delta^{\dot{\alpha}}_{\dot{\delta}}\delta^{\dot{\gamma}}_{\dot{\beta}}) &\quad (B.34)
\end{aligned}$$

Also useful are the following identities:

$$(\sigma^{\rho\sigma}\sigma^{\mu\nu})^{\alpha}_{\beta}v_{\rho\sigma}v_{\mu\nu} = -\frac{1}{2}\delta^{\alpha}_{\beta}v_{\mu\nu}\hat{v}^{\mu\nu} \qquad (\bar{\sigma}^{\rho\sigma}\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}}v_{\rho\sigma}v_{\mu\nu} = -\frac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}}v_{\mu\nu}\hat{v}^{\dagger\mu\nu} \tag{B.35}$$

$$\begin{aligned}
\sigma^{\rho\sigma}\sigma^{\mu}v_{\rho\sigma} &= \hat{v}^{\mu\nu}\sigma_{\nu}\\
\bar{\sigma}^{\mu}\sigma^{\rho\sigma}v_{\rho\sigma} &= -\hat{v}^{\mu\nu}\bar{\sigma}_{\nu}\\
\sigma^{\mu}\bar{\sigma}^{\rho\sigma}v_{\rho\sigma} &= -\hat{v}^{\dagger\mu\nu}\sigma_{\nu}\\
\bar{\sigma}^{\rho\sigma}\bar{\sigma}^{\mu}v_{\rho\sigma} &= \hat{v}^{\dagger\mu\nu}\bar{\sigma}_{\nu}
\end{aligned} \tag{B.36}$$

where

$$\hat{v}^{\mu\nu} = v^{\mu\nu} + \frac{i}{2}v^{*\mu\nu} \qquad \hat{v}^{\dagger\mu\nu} = v^{\mu\nu} - \frac{i}{2}v^{*\mu\nu} \tag{B.37}$$

and $v^{\mu\nu} = -v^{\nu\mu}$.

B.3 A typical calculation

We present here an example of a typical calculation encountered during the lengthy computations we dealt with.
Often we have to reduce expressions of the form

$$\lambda\sigma^{0i}\psi\,\chi\sigma_{0i}\varphi \tag{B.38}$$

In order to do that first we have to write in the spinor indices, then extract the matrices being careful about the position of the spinors involved. Thus

the expression above becomes

$$\lambda^\alpha \psi_\beta \chi^\gamma \varphi_\delta \;\; \sigma^{0i\beta}_{\alpha} \sigma^{\delta}_{0i\gamma} \tag{B.39}$$

Then we use the definition (B.6) to write the product of the matrices as

$$\frac{1}{4}(\sigma^0_{\alpha\dot{\alpha}}\bar{\sigma}^{i\dot{\alpha}\beta} - \sigma^i_{\alpha\dot{\alpha}}\bar{\sigma}^{0\dot{\alpha}\beta})\sigma^{\delta}_{0i\gamma} \tag{B.40}$$

using the properties (B.28) and (B.29) this becomes

$$\frac{1}{8}\Big((\sigma^0\bar{\sigma}_0)^{\epsilon}_{\alpha}\epsilon_{\gamma\epsilon}\epsilon^{\delta\beta} + (\sigma^0\bar{\sigma}_0)^{\delta}_{\alpha}\delta^{\beta}_{\gamma} - (\sigma_0\bar{\sigma}^0)^{\beta}_{\epsilon}\epsilon_{\alpha\gamma}\epsilon^{\delta\epsilon} + (\sigma_0\bar{\sigma}^0)^{\beta}_{\gamma}\delta^{\delta}_{\alpha}\Big) \tag{B.41}$$

using $(\sigma^0\bar{\sigma}_0)^{\beta}_{\alpha} = -\delta^{\beta}_{\alpha}$ we obtain

$$\frac{1}{8}\Big(- \epsilon_{\gamma\alpha}\epsilon^{\delta\beta} - \delta^{\delta}_{\alpha}\delta^{\beta}_{\gamma} + \epsilon_{\alpha\gamma}\epsilon^{\delta\beta} - \delta^{\beta}_{\gamma}\delta^{\delta}_{\alpha}\Big) = -\frac{1}{4}\Big(\epsilon_{\gamma\alpha}\epsilon^{\delta\beta} + \delta^{\delta}_{\alpha}\delta^{\beta}_{\gamma}\Big) \tag{B.42}$$

When we substitute this back in (B.39), pay attention to the summation conventions and commute the spinors we end up with

$$-\frac{1}{4}(\psi\varphi\,\lambda\chi - \psi\chi\,\lambda\varphi) \tag{B.43}$$

In the case where $\varphi \equiv \lambda$ we can reduce the expression even more using the Fierz identities given in (B.13) in [9]. In fact we can write

$$\psi^\alpha\lambda_\alpha\,\lambda^\beta\chi_\beta = -\frac{1}{2}\delta^{\beta}_{\alpha}\psi^\alpha\chi_\beta\,\lambda^2 = -\frac{1}{2}\psi\chi\,\lambda^2 \tag{B.44}$$

Thus for $\varphi \equiv \lambda$ the expression (B.39) can be reduced to

$$\frac{3}{8}\psi\chi\,\lambda^2 \tag{B.45}$$

B.4 Derivation with respect to a grassmanian variable

The derivative $\frac{\delta}{\delta\psi}$ is a grassmanian variable therefore anti commutes. From the general rule $\partial_\mu = \eta_{\nu\mu}\partial^\nu$ it follows that the indices have to be raised

and lowered with the opposite metric tensor with respect to the standard convention

$$\frac{\delta}{\delta\psi^{\alpha}} = \epsilon_{\beta\alpha}\frac{\delta}{\delta\psi_{\beta}} \tag{B.46}$$

This is crucial to get the signs right, for instance:

$$\frac{\delta}{\delta\psi^{\gamma}}(\psi\psi) = \frac{\delta}{\delta\psi^{\gamma}}(\psi^{\alpha}\psi_{\alpha}) = \epsilon_{\alpha\beta}\frac{\delta}{\delta\psi^{\gamma}}(\psi^{\alpha}\psi^{\beta}) = \epsilon_{\alpha\beta}(\delta^{\alpha}_{\gamma}\psi^{\beta} - \psi^{\alpha}\delta^{\beta}_{\gamma}) = +2\psi_{\gamma} \tag{B.47}$$

and

$$\frac{\delta}{\delta\psi_{\gamma}}(\psi\psi) = \epsilon^{\beta\gamma}\frac{\delta}{\delta\psi^{\beta}}(\psi^{\alpha}\psi_{\alpha}) = \epsilon^{\beta\gamma}(2\psi_{\beta}) = -2\psi^{\gamma} \tag{B.48}$$

Similarly for dotted indices

$$\frac{\delta}{\delta\bar{\psi}_{\dot{\gamma}}}(\bar{\psi}\bar{\psi}) = \frac{\delta}{\delta\bar{\psi}_{\dot{\gamma}}}(\bar{\psi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}) = +2\bar{\psi}^{\dot{\gamma}} \quad \frac{\delta}{\delta\bar{\psi}^{\dot{\gamma}}}(\bar{\psi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}) = -2\bar{\psi}_{\dot{\gamma}} \tag{B.49}$$

Form here it is clear why the momenta have to be treated with the opposite convention.

Appendix C

Graded Poisson brackets

We deal with c-number valued fields, i.e. non operator, in the classical as well as effective case. Therefore the Susy algebra has to be implemented via graded Poisson brackets, namely with Poisson brackets $\{,\}_-$ and anti-brackets $\{,\}_+$. We define the following equal time Poisson (anti) brackets[1]

$$\{B_1(x), B_2(y)\}_- \equiv \int d^3z \Big(\frac{\delta B_1(x)}{\delta \Phi(z)} \frac{\delta B_2(y)}{\delta \Pi(z)} - \frac{\delta B_2(y)}{\delta \Phi(z)} \frac{\delta B_1(x)}{\delta \Pi(z)} \Big) \quad \text{(C.1)}$$

$$\{B(x), F(y)\}_- \equiv \int d^3z \Big(\frac{\delta B(x)}{\delta \Phi(z)} \frac{\delta F(y)}{\delta \Pi(z)} - \frac{\delta F(y)}{\delta \Phi(z)} \frac{\delta B(x)}{\delta \Pi(z)} \Big) \quad \text{(C.2)}$$

$$\{F_1(x), F_2(y)\}_+ \equiv \int d^3z \Big(\frac{\delta F_1(x)}{\delta \Phi(z)} \frac{\delta F_2(y)}{\delta \Pi(z)} + \frac{\delta F_2(y)}{\delta \Phi(z)} \frac{\delta F_1(x)}{\delta \Pi(z)} \Big) \quad \text{(C.3)}$$

where the B's are bosonic and the F's fermionic variables and Φ and Π span the whole phase space.

Form this definition it follows that the properties of the graded Poisson

[1] Following a nice argument given by Dirac [20], this definition leads to the quantum *anti*-commutator for two fermions. The original argument relates classical Poisson brackets to the commutator $[B_1(x), B_2(y)]_- \to i\hbar\{B_1(x), B_2(y)\}_-$, where the B's stand for bosonic variables. The generalization to fermions is naturally given by $[F_1(x), F_2(y)]_+ \to i\hbar\{F_1(x), F_2(y)\}_+$ (we shall use the natural units $\hbar = c = 1$), where the F's are fermionic variables.

brackets are the same as for standard commutators and anti-commutators

$$\{B_1, B_2\}_- = -\{B_2, B_1\}_- \quad \{B, F\}_- = -\{F, B\}_- \quad \{F_1, F_2\}_+ = +\{F_2, F_1\}_+ \tag{C.4}$$

Let us notice that only a formal algebraic meaning can be associated to the Poisson anti-bracket of two fermions, since there is no physical meaning for a *classical* fermion.

The Susy algebra (1.10)-(1.12) is modified by a factor i due to the relation[2] $[,]_\pm \to i\{,\}_\pm$.

The canonical equal-time Poisson brackets for a Lagrangian with ϕ and ψ as boson and fermionic fields respectively are given by the usual relations

$$\{\phi(\vec{x}, t)\ ,\ \pi_\phi(\vec{y}, t)\}_- = \delta^{(3)}(\vec{x} - \vec{y}) \quad \{\psi^\alpha(\vec{x}, t)\ ,\ \pi_{\psi\beta}(\vec{y}, t)\}_+ = \delta^\alpha_\beta \delta^{(3)}(\vec{x} - \vec{y}) \tag{C.5}$$

and

$$\{\phi\ ,\ \phi\}_- = \{\psi\ ,\psi\ \}_+ = \{\pi_\phi\ ,\ \pi_\phi\}_- = \{\pi_\psi\ ,\pi_\psi\ \}_+ = \{\phi\ ,\ \pi_\psi\}_- = \{\psi\ ,\ \pi_\phi\}_- = 0 \tag{C.6}$$

The same structure survives at the effective level even if a great deal of care is required.

Note that

$$\{\psi_\alpha\ ,\ \pi^\beta_\psi\}_+ = \delta^\beta_\alpha \quad \text{and} \quad \{\psi^\alpha\ ,\ \pi_{\psi\beta}\}_+ = \delta^\alpha_\beta \tag{C.7}$$

are compatible iff $\pi_\alpha = -\epsilon_{\alpha\beta}\pi^\beta$ which is the convention explained in Appendix B. Note also that we impose

$$\{\psi_\alpha\ ,\ \pi_{\psi\beta}\}_+ = \epsilon_{\alpha\beta} = \{\pi_{\psi\alpha}\ ,\ \psi_\beta\}_+ \tag{C.8}$$

We use the graded Poisson brackets in the same spirit of derivatives, thus even if some of the variables involved are not dynamical we have to commute

[2] See previous Note.

them. For instance $\{\epsilon\psi, \pi_\psi\}_- = \epsilon\{\psi, \pi_\psi\}_+ - \{\epsilon, \pi_\psi\}_+\psi = \epsilon\{\psi, \pi_\psi\}_+$ where ψ , π_ψ are dynamical and ϵ is just a grassmanian parameter.

Useful identities:

$$\{B_1\ ,\ B_2B_3\}_- = \{B_1, B_2\}_-B_3 + B_2\{B_1, B_3\}_- \tag{C.9}$$

$$\{B_1B_2\ ,\ B_3\}_- = B_1\{B_2, B_3\}_- + \{B_1, B_3\}_-B_2 \tag{C.10}$$

$$\{F_1\ ,\ F_2F_3\}_- = \{F_1, F_2\}_+F_3 - F_2\{F_1, F_3\}_+ \tag{C.11}$$

$$\{F_1F_2\ ,\ F_3\}_- = F_1\{F_2, F_3\}_+ - \{F_1, F_3\}_+F_2 \tag{C.12}$$

$$\{F_1F_2\ ,\ B\}_- = F_1\{F_2, B\}_- + \{F_1, B\}_-F_2 \tag{C.13}$$

$$\{B\ ,\ F_3F_4\}_- = \{B, F_3\}_-F_4 + F_3\{B, F_4\}_- \tag{C.14}$$

$$\{F_1F_2\ ,\ F_3F_4\}_- = F_1\{F_2, F_3\}_+F_4 - F_1\{F_2, F_4\}_+F_3 \tag{C.15}$$

$$-F_2\{F_1, F_3\}_+F_4 + F_2\{F_1, F_4\}_+F_3 \tag{C.16}$$

where the B's and the F's are bosonic and fermionic variables respectively.

Appendix D

Computation of the effective V_μ

We first notice that, by varying off-shell the Lagrangian (3.66) (the one not integrated by parts) under the Susy transformations given in (3.2)-(3.7), there is no mixing of the $\mathcal{F}$ terms with the $\mathcal{F}^\dagger$ terms. As explained in Chapter 3, the structure of the Lagrangian is

$$2i\mathcal{L} = -\mathcal{F}''[2B+2F] + \mathcal{F}'''[1B2F] + \mathcal{F}''''[4F] \tag{D.1}$$
$$+ \mathcal{F}^{\dagger''}[2B+2F]^\dagger - \mathcal{F}^{\dagger'''}[1B2F]^\dagger - \mathcal{F}^{\dagger''''}[4F]^\dagger \tag{D.2}$$

where B and F stand for bosonic and fermionic variables, respectively.
For instance, if we vary the $\mathcal{F}$ terms under δ_1 we have $(\delta_1\mathcal{F}'''')[3] = 0$, whereas the other terms combine as follows

$$\mathcal{F}''''\delta_1[4F] \sim \mathcal{F}''''(1B3F) \quad \text{with} \quad (\delta_1\mathcal{F}''')[1B2F] \sim \mathcal{F}''''(1B3F)$$
$$\mathcal{F}'''\delta_1[1B2F] \sim \mathcal{F}'''(2B1F+3F) \quad \text{with} \quad (\delta_1\mathcal{F}'')[2B+2F] \sim \mathcal{F}'''(2B1F+3F)$$

Finally there are terms $\mathcal{F}''\delta_1[2B+2F]$, the naive generalization of the classical V_1^μ. The aim is to write these quantities as one single total divergence

and express it in terms of momenta and variations of the fields, that we write down again here

$$\pi^\mu_A = -\mathcal{I}\partial^\mu A^\dagger \qquad \pi^\mu_{A^\dagger} = (\pi^\mu_A)^\dagger \tag{D.3}$$

$$\Pi^{\mu\nu} = -\frac{1}{2i}(\mathcal{F}''\hat{v}^{\mu\nu} - \mathcal{F}^{\dagger\prime\prime}\hat{v}^{\dagger\mu\nu}) + \frac{1}{i\sqrt{2}}(\mathcal{F}'''\lambda\sigma^{\mu\nu}\psi - \mathcal{F}^{\dagger\prime\prime\prime}\bar{\lambda}\bar{\sigma}^{\mu\nu}\bar{\psi}) \tag{D.4}$$

$$(\pi^\mu_{\bar{\psi}})_{\dot{\alpha}} = \frac{1}{2}\mathcal{F}''\psi^\alpha\sigma^\mu_{\alpha\dot{\alpha}} \quad (\pi^\mu_\psi)^\alpha = -\frac{1}{2}\mathcal{F}^{\dagger\prime\prime}\bar{\psi}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha} \tag{D.5}$$

$$(\pi^\mu_{\bar{\lambda}})_{\dot{\alpha}} = \frac{1}{2}\mathcal{F}''\lambda^\alpha\sigma^\mu_{\alpha\dot{\alpha}} \quad (\pi^\mu_\lambda)^\alpha = -\frac{1}{2}\mathcal{F}^{\dagger\prime\prime}\bar{\lambda}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha} \tag{D.6}$$

$$\begin{aligned} \delta_1 A &= \sqrt{2}\epsilon_1\psi \\ \delta_1\psi^\alpha &= \sqrt{2}\epsilon_1^\alpha E \\ \delta_1 E &= 0 \end{aligned} \tag{D.7}$$

$$\begin{aligned} \delta_1 E^\dagger &= i\sqrt{2}\epsilon_1 \ \partial\!\!\!/\bar{\psi} \\ \delta_1\bar{\psi}_{\dot{\alpha}} &= -i\sqrt{2}\epsilon_1^\alpha \ \partial\!\!\!/_{\alpha\dot{\alpha}}A^\dagger \\ \delta_1 A^\dagger &= 0 \end{aligned} \tag{D.8}$$

$$\begin{aligned} \delta_1\lambda^\alpha &= -\epsilon_1^\beta(\sigma^{\mu\nu}{}_\beta{}^\alpha v_{\mu\nu} - i\delta^\alpha_\beta D) \\ \delta_1 v^\mu &= i\epsilon_1\sigma^\mu\bar{\lambda} \quad \delta_1 D = -\epsilon_1 \ \partial\!\!\!/\bar{\lambda} \\ \delta_1\bar{\lambda}_{\dot{\alpha}} &= 0 \end{aligned} \tag{D.9}$$

This computation is by no means easy. It is matter of

- identifying similar terms and compare them
- use partial integration cleverly: never throw away surface terms!
- use extensively Fierz identities and spinor algebra

We have found by direct computation V_1^μ and $\bar{V}_1^\mu$. Of course the first one has been the most difficult to find, since if one understands how to proceed in the first case, the other cases become only lengthy checks. We do not have the space here to explicitly show all the details. What we want to show explicitly in this Appendix, is only the simplest part of the computation of V_1^μ, namely the contribution coming from the $\mathcal{F}$ terms.

Let us apply the scheme discussed above. First we consider the $\mathcal{F}''''$ type of terms. If we find contributions from these terms we know that they cannot be canceled by terms coming from the rigid current N_μ and there is no hope to rearrange them in the form of on-shell dummy fields (they only contain $\mathcal{F}'''$ type of terms). This would then be a signal that by commuting the charges we could have contributions that would spoil the SW mass formula. What we find is that the terms

$$\begin{aligned}\mathcal{F}''''\delta_1[4F] &= \mathcal{F}''''\frac{1}{2}[(\delta_1\psi)\psi\lambda^2 + \psi^2(\delta_1\lambda)\lambda] \\ &= \mathcal{F}''''\frac{1}{2}[\sqrt{2}\epsilon_1\psi\lambda^2 E - \epsilon_1\sigma^{\mu\nu}\lambda v_{\mu\nu}\psi^2 + i\epsilon_1\lambda\psi^2 D] \end{aligned} \tag{D.10}$$

summed to the terms

$$(\delta_1\mathcal{F}''')[1B2F] = \mathcal{F}''''[\epsilon_1\psi\lambda\sigma^{\mu\nu}\psi v_{\mu\nu} - \frac{1}{\sqrt{2}}E\epsilon_1\psi\lambda^2 + iD\epsilon_1\psi\psi\lambda] \tag{D.11}$$

fortunately give zero.

Let us then move to the next level, the $\mathcal{F}'''$ terms. In principle these terms can be present, since they appear in the expression of the on-shell dummy fields. We find that the terms

$$\begin{aligned}\mathcal{F}'''[(\delta_1 1B)2F] &= \mathcal{F}'''[\frac{1}{\sqrt{2}}\lambda\sigma^{\mu\nu}\psi\delta_1 v_{\mu\nu} - \frac{1}{2}(\delta_1 E^\dagger)\psi\psi + \frac{i}{\sqrt{2}}(\delta_1 D)\psi\lambda] \\ &= \mathcal{F}'''[\frac{1}{\sqrt{2}}\lambda\sigma^{\mu\nu}\psi 2i\epsilon_1\sigma_\nu\partial_\mu\bar{\lambda} - \frac{i}{\sqrt{2}}\epsilon_1\ \partial\!\!\!/\bar{\psi}\psi\psi - \frac{i}{\sqrt{2}}\epsilon_1\ \partial\!\!\!/\bar{\lambda}\psi\lambda] \\ &= \frac{i}{\sqrt{2}}\mathcal{F}'''(2\lambda\ \partial\!\!\!/\bar{\lambda}\epsilon_1\psi - \epsilon_1\ \partial\!\!\!/\bar{\psi}\psi\psi) \end{aligned} \tag{D.12}$$

summed to the terms

$$-(\delta_1\mathcal{F}'')[2F] = -i\sqrt{2}\mathcal{F}'''\lambda \not{\partial}\bar{\lambda}\epsilon_1\psi - i\sqrt{2}\mathcal{F}'''\psi \not{\partial}\bar{\psi}\epsilon_1\psi) \tag{D.13}$$

again give zero.

What is left are the other $\mathcal{F}'''$ terms and the $\mathcal{F}''$ terms. There we find

$$\begin{aligned}
-\mathcal{F}''\delta_1[2B+2F] &= -\mathcal{F}''[(\partial_\mu A^\dagger)\partial^\mu(\delta_1 A) + \frac{1}{2}(\delta_1 v_{\mu\nu})v^{\mu\nu} + \frac{i}{4}(\delta_1 v_{\mu\nu})v^{*\mu\nu} \\
&\quad +i(\delta_1\psi)\not{\partial}\bar{\psi} + i\psi\not{\partial}(\delta_1\bar{\psi}) + i(\delta_1\lambda)\not{\partial}\bar{\lambda} - E\delta_1 E^\dagger - D\delta_1 D] \\
&= -\mathcal{F}''[(\partial_\mu A^\dagger)\partial^\mu(\sqrt{2}\epsilon_1\psi) + (i\epsilon_1\sigma_\nu(\partial_\mu\bar{\lambda}))v^{\mu\nu} \\
&\quad -\frac{1}{2}(i\epsilon_1\sigma_\nu(\partial_\mu\bar{\lambda}))v^{*\mu\nu} + i\sqrt{2}\epsilon_1\not{\partial}\bar{\psi}E - \sqrt{2}\psi^\alpha(\not{\partial}_{\alpha\dot{\alpha}}\not{\partial}^{\dot{\alpha}\beta}A^\dagger)\epsilon_{1\beta} \\
&\quad -i\epsilon_1^\beta(\sigma^{\mu\nu}{}_\beta{}^\alpha v_{\mu\nu} - i\delta_\beta^\alpha D)\not{\partial}_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} \\
&\quad -i\sqrt{2}\epsilon_1\not{\partial}\bar{\psi}E - D\epsilon_1\not{\partial}\bar{\lambda}] \\
&= -\mathcal{F}''[\sqrt{2}\epsilon_1(\partial^\mu\psi)\partial_\mu A^\dagger + \sqrt{2}\epsilon_1\psi\Box A^\dagger] \\
&= -\mathcal{F}''\partial^\mu[\sqrt{2}\epsilon_1\psi\partial_\mu A^\dagger]
\end{aligned} \tag{D.14}$$

Thus we find the first non zero contribution. Let us note that this term would already be a total divergence if we impose the classical limit $\mathcal{F}'' \to \tau$. Thus we can guess that the $\mathcal{F}'''$ terms have to combine to give the quantum piece missing in order to built up a total divergence when summed to the terms (D.14). We find that

$$-\delta_1\mathcal{F}''[2B] = \mathcal{F}'''\sqrt{2}\epsilon_1\psi[-\partial_\mu A^\dagger\partial^\mu A - \frac{1}{4}v_{\mu\nu}\hat{v}^{\mu\nu} + EE^\dagger + \frac{1}{2}D^2]$$

summed to

$$\begin{aligned}
\mathcal{F}'''[1B(\delta_1 2F)] &= \mathcal{F}'''[\frac{1}{\sqrt{2}}(\delta_1\lambda)\sigma^{\mu\nu}\psi v_{\mu\nu} + \frac{1}{\sqrt{2}}\lambda\sigma^{\mu\nu}(\delta_1\psi)v_{\mu\nu} \\
&\quad -\frac{1}{2}(E^\dagger\psi\delta_1\psi + E\lambda\delta_1\lambda) + \frac{i}{\sqrt{2}}D(\delta_1\psi)\lambda + \frac{i}{\sqrt{2}}D\psi\delta_1\lambda] \\
&= \mathcal{F}'''[-\frac{1}{\sqrt{2}}\epsilon_1\sigma^{\mu\nu}\sigma^{\rho\sigma}\psi v_{\mu\nu}v_{\rho\sigma} + \frac{i}{\sqrt{2}}\epsilon_1\sigma^{\mu\nu}\psi v_{\mu\nu}D
\end{aligned}$$

$$
\begin{aligned}
& +\lambda\sigma^{\mu\nu}\epsilon_1 v_{\mu\nu}E - E^\dagger E\sqrt{2}\epsilon_1\psi + E\epsilon_1\sigma^{\mu\nu}\lambda v_{\mu\nu}) \\
& -i\epsilon_1\lambda DE + iDE\epsilon_1\lambda - \frac{i}{\sqrt{2}}D\epsilon_1\sigma^{\mu\nu}v_{\mu\nu}\psi - \frac{1}{\sqrt{2}}D^2\epsilon_1\psi] \\
= \; & -\mathcal{F}'''[\frac{1}{\sqrt{2}}\epsilon_1\sigma^{\mu\nu}\sigma^{\rho\sigma}v_{\mu\nu}v_{\rho\sigma}\psi + E^\dagger E\sqrt{2}\epsilon_1\psi + \frac{1}{\sqrt{2}}D^2\epsilon_1\psi]
\end{aligned}
$$

give

$$
(\partial^\mu\mathcal{F}'')[-\sqrt{2}\epsilon_1\psi\partial_\mu A^\dagger] \tag{D.15}
$$

Collecting the two contributions (D.14) and (D.15) we end up with the wanted total divergence

$$
\begin{aligned}
& \mathcal{F}''\partial^\mu[-\sqrt{2}\epsilon_1\psi\partial_\mu A^\dagger] + (\partial^\mu\mathcal{F}'')[-\sqrt{2}\epsilon_1\psi\partial_\mu A^\dagger] \\
= \; & \partial^\mu(-\mathcal{F}''\sqrt{2}\epsilon_1\psi\partial_\mu A^\dagger) \\
= \; & \partial^\mu(\frac{\mathcal{F}''}{\mathcal{I}}\delta_1 A\pi_A^\mu)
\end{aligned} \tag{D.16}
$$

where the definitions of momenta and the Susy transformations were used. More labour is needed for the $\mathcal{F}^\dagger$ terms. We only give the result of that computation here. We have

$$
\begin{aligned}
& \partial_\mu[\mathcal{F}''^\dagger\sqrt{2}\epsilon_1\psi\partial^\mu A^\dagger + i\mathcal{F}''^\dagger\epsilon_1\sigma_\nu\bar{\lambda}\hat{v}^{\mu\nu\dagger} + \mathcal{F}''^\dagger\epsilon_1\sigma^\mu\bar{\lambda}D \\
& +\mathcal{F}''^\dagger\sqrt{2}\epsilon_1\sigma^\nu\sigma^\mu\psi\partial_\nu A^\dagger + i\sqrt{2}\mathcal{F}''^\dagger\bar{\psi}\bar{\sigma}^\mu\epsilon_1 E + \frac{i}{\sqrt{2}}\mathcal{F}'''^\dagger\epsilon_1\sigma^\mu\bar{\psi}\lambda^2]
\end{aligned}
$$

Using the definitions of the non canonical momenta and the Susy transformations of the fields, these terms can be recast into the following form

$$
\begin{aligned}
& \partial_\mu[-\frac{\mathcal{F}''^\dagger}{\mathcal{I}}\delta_1 A\pi_A^\mu + 2i\frac{\mathcal{F}''^\dagger}{\mathcal{F}''}\delta_1\bar{\psi}\pi_{\bar{\psi}}^\mu + 2i\delta_1\lambda\pi_\lambda^\mu + \mathcal{F}^{\dagger\prime\prime}\epsilon_1\sigma_\nu\bar{\lambda}v^{*\mu\nu} \\
& \qquad +2i\delta_1\psi\pi_\psi^\mu + \frac{i}{\sqrt{2}}\mathcal{F}'''^\dagger\epsilon_1\sigma^\mu\bar{\psi}\lambda^2]
\end{aligned} \tag{D.17}
$$

Summing up the terms (D.16) and (D.17) and dividing by $2i$ we obtain the final expression

$$
\begin{aligned}
V_1^\mu \; = \; & \delta_1 A\pi_A^\mu + \frac{\mathcal{F}^{\dagger\prime\prime}}{\mathcal{F}''}\delta_1\bar{\psi}\pi_{\bar{\psi}}^\mu + \delta_1\psi\pi_\psi^\mu + \delta_1\lambda\pi_\lambda^\mu \\
& +\frac{1}{2i}\mathcal{F}^{\dagger\prime\prime}\epsilon_1\sigma_\nu\bar{\lambda}v^{*\mu\nu} + \frac{1}{2\sqrt{2}}\mathcal{F}^{\dagger\prime\prime\prime}\epsilon_1\sigma^\mu\bar{\psi}\bar{\lambda}^2
\end{aligned} \tag{D.18}
$$

As explained in Chapter 3 this form is not canonical and has to be modified according to the rules given there.

Appendix E

Transformations from the U(1) effective charges

In this Appendix we shall complete the proof given in Chapter 3 that our U(1) effective charges correctly generate the Susy transformations.

E.1 $\Delta_1\lambda$ from Q_1^I

On the one hand

$$\begin{aligned}\Delta_1\pi^I_{\bar\lambda\dot\alpha}(x) &\equiv \{\epsilon_1 Q_1\ ,\ \pi^I_{\bar\lambda\dot\alpha}(x)\}_- \\ &= \int d^3y(\Pi^i(y)\{\delta_1 v_i(y)\ ,\ \pi^I_{\bar\lambda\dot\alpha}(x)\}_- \\ &\quad -\frac{1}{2i}\mathcal{F}^{\dagger\prime\prime}(y)v^{*0i}(y)\{\epsilon_1\sigma_i\bar\lambda(y)\ ,\ \pi^I_{\bar\lambda\dot\alpha}(x)\}_- \\ &\quad -\frac{1}{2\sqrt{2}}\mathcal{F}^{\dagger\prime\prime\prime}(y)\epsilon_1\sigma^0\bar\psi(y)\{\bar\lambda^2(y)\ ,\ \pi^I_{\bar\lambda\dot\alpha}(x)\}_-)\end{aligned} \tag{E.1}$$

On the other hand

$$\Delta_1\pi^I_{\bar\lambda\dot\alpha} = \frac{1}{\sqrt{2}}\mathcal{F}'''\epsilon_1\psi\lambda^\alpha\sigma^0_{\alpha\dot\alpha} + i\mathcal{I}\sigma^0_{\alpha\dot\alpha}\Delta_1\lambda^\alpha \tag{E.2}$$

Equating the two expressions, writing explicitly Π^i and collecting the terms according to the order of the derivative of $\mathcal{F}$ we have

$$\begin{aligned} i\mathcal{I}\sigma^0_{\alpha\dot{\alpha}}\Delta_1\lambda^\alpha &= -\frac{1}{2}(\mathcal{F}''\hat{v}^{0i} - \mathcal{F}^{\dagger''}\hat{v}^{\dagger 0i})\epsilon_1^\alpha\sigma_{i\alpha\dot{\alpha}} + \frac{i}{2}\mathcal{F}^{\dagger''}v^{*0i}\epsilon_1^\alpha\sigma_{i\alpha\dot{\alpha}} \\ &\quad +\frac{1}{\sqrt{2}}\mathcal{F}'''\lambda\sigma^{0i}\psi\epsilon_1^\alpha\sigma_{i\alpha\dot{\alpha}} - \frac{1}{\sqrt{2}}\mathcal{F}'''\epsilon_1\psi\lambda^\alpha\sigma^0_{\alpha\dot{\alpha}} \\ &\quad -\frac{1}{\sqrt{2}}\mathcal{F}^{\dagger'''}\bar{\lambda}\bar{\sigma}^{0i}\bar{\psi}\epsilon_1^\alpha\sigma_{i\alpha\dot{\alpha}} - \frac{1}{\sqrt{2}}\mathcal{F}^{\dagger'''}\epsilon_1\sigma^0\bar{\psi}\bar{\lambda}_{\dot{\alpha}} \end{aligned} \tag{E.3}$$

The terms are arranged such that in the first column there are terms from Π^i and in the second the others. Now we notice that the terms in the first line should combine to give the term proportional to $v_{\mu\nu}$ and the other two lines should combine to give the term proportional to D^{on} in $\delta_1\lambda$. First line:

$$-\frac{1}{2}(\mathcal{F}'' - \mathcal{F}^{\dagger''})\hat{v}^{0i}\epsilon_1^\alpha\sigma_{i\alpha\dot{\alpha}} = -i\mathcal{I}(\epsilon_1\sigma^{\mu\nu}\sigma^0)_{\dot{\alpha}}v_{\mu\nu} \tag{E.4}$$

where the identity $\hat{v}^{0i}\sigma_{i\alpha\dot{\alpha}} = (\sigma^{\mu\nu}\sigma^0)_{\alpha\dot{\alpha}}v_{\mu\nu}$ was used (see relative appendix).
Second line:
The first term in the second line

$$\frac{1}{\sqrt{2}}\mathcal{F}'''\lambda^\beta(\sigma^{0i})_\beta^\gamma\psi_\gamma\epsilon_1^\alpha\sigma_{i\alpha\dot{\alpha}} = \frac{1}{2\sqrt{2}}\mathcal{F}'''(\epsilon_1\psi\lambda^\alpha\sigma^0_{\alpha\dot{\alpha}} - \epsilon_1\lambda\psi^\alpha\sigma^0_{\alpha\dot{\alpha}}) \tag{E.5}$$

where the identity

$$\sigma^{0i}{}_\beta{}^\gamma\sigma_{i\,\alpha\dot{\alpha}} = \frac{1}{2}(\sigma^0_{\delta\dot{\alpha}}\epsilon_{\alpha\beta}\epsilon^{\gamma\delta} - \sigma^0_{\beta\dot{\alpha}}\delta_\alpha^\gamma) \tag{E.6}$$

was used (see Appendix B). Combined with the second term in the same line we have

$$-\frac{1}{2\sqrt{2}}\mathcal{F}'''(\epsilon_1\psi\lambda^\alpha\sigma^0_{\alpha\dot{\alpha}} + \epsilon_1\lambda\psi^\alpha\sigma^0_{\alpha\dot{\alpha}}) = \frac{1}{2\sqrt{2}}\mathcal{F}'''\psi\lambda\epsilon_1^\alpha\sigma^0_{\alpha\dot{\alpha}} \tag{E.7}$$

where we used the Fierz identity $\lambda_\beta\psi^\alpha = -\psi_\beta\lambda^\alpha - \delta_\beta^\alpha\psi\lambda$. Third line:
Using similar identities (see the appendix) we can write the first term in the third line as follows

$$\frac{1}{2\sqrt{2}}\mathcal{F}^{\dagger'''}(\epsilon_1\sigma^0\bar{\psi}\bar{\lambda}_{\dot{\alpha}} - \epsilon_1\sigma^0\bar{\lambda}\bar{\psi}_{\dot{\alpha}}) \tag{E.8}$$

which combined with the second term in the same line gives

$$\frac{1}{2\sqrt{2}}\mathcal{F}^{\dagger'''}\bar{\psi}\bar{\lambda}\epsilon_1^\alpha\sigma^0_{\alpha\dot{\alpha}} \tag{E.9}$$

Collecting the terms in (E.4), (E.7) and (E.9) we have

$$i\mathcal{I}\sigma^0_{\alpha\dot{\alpha}}\Delta_1\lambda^\alpha = -i\mathcal{I}(\epsilon_1\sigma^{\mu\nu}\sigma^0)_{\dot{\alpha}}v_{\mu\nu} + \frac{1}{2\sqrt{2}}\mathcal{F}'''\psi\lambda\epsilon_1^\alpha\sigma^0_{\alpha\dot{\alpha}} + \frac{1}{2\sqrt{2}}\mathcal{F}^{\dagger'''}\bar{\psi}\bar{\lambda}\epsilon_1^\alpha\sigma^0_{\alpha\dot{\alpha}} \tag{E.10}$$

which eventually gives the wanted expression (3.118)

E.2 $\Delta_2\psi$ from Q_2^I

On the one hand

$$\begin{aligned}
\Delta_2\pi^I_{\bar{\psi}\dot{\alpha}}(x) &\equiv \{\epsilon_2 Q_2\ ,\ \pi^I_{\bar{\psi}\dot{\alpha}}(x)\}_- \\
&= \int d^3y(\Pi^i(y)\{\delta_2 v_i(y)\ ,\ \pi^I_{\bar{\psi}\dot{\alpha}}(x)\}_- \\
&\quad -\frac{1}{2i}\mathcal{F}^{\dagger''}(y)v^{*0i}(y)\{\epsilon_2\sigma_i\bar{\psi}(y)\ ,\ \pi^I_{\bar{\psi}\dot{\alpha}}(x)\}_- \\
&\quad +\frac{1}{2\sqrt{2}}\mathcal{F}^{\dagger'''}(y)\epsilon_2\sigma^0\bar{\lambda}(y)\{\bar{\psi}^2(y)\ ,\ \pi^I_{\bar{\psi}\dot{\alpha}}(x)\}_-)
\end{aligned} \tag{E.11}$$

On the other hand

$$\Delta_2\pi^I_{\bar{\psi}\dot{\alpha}} = -\frac{1}{\sqrt{2}}\mathcal{F}'''\epsilon_2\lambda\psi^\alpha\sigma^0_{\alpha\dot{\alpha}} + i\mathcal{I}\sigma^0_{\alpha\dot{\alpha}}\Delta_2\psi^\alpha \tag{E.12}$$

Equating the two expressions, writing explicitly Π^i and collecting the terms according to the order of the derivative of $\mathcal{F}$ we have

$$\begin{aligned}
i\mathcal{I}\sigma^0_{\alpha\dot{\alpha}}\Delta_2\psi^\alpha &= -\frac{1}{2}(\mathcal{F}''\hat{v}^{0i} - \mathcal{F}^{\dagger''}\hat{v}^{\dagger 0i})\epsilon_2^\alpha\sigma_{i\alpha\dot{\alpha}} + \frac{i}{2}\mathcal{F}^{\dagger''}v^{*0i}\epsilon_2^\alpha\sigma_{i\alpha\dot{\alpha}} \\
&\quad -\frac{1}{\sqrt{2}}\mathcal{F}'''\psi\sigma^{0i}\lambda\epsilon_2^\alpha\sigma_{i\alpha\dot{\alpha}} + \frac{1}{\sqrt{2}}\mathcal{F}'''\epsilon_2\lambda\psi^\alpha\sigma^0_{\alpha\dot{\alpha}} \\
&\quad +\frac{1}{\sqrt{2}}\mathcal{F}^{\dagger'''}\bar{\psi}\bar{\sigma}^{0i}\bar{\lambda}\epsilon_2^\alpha\sigma_{i\alpha\dot{\alpha}} + \frac{1}{\sqrt{2}}\mathcal{F}^{\dagger'''}\epsilon_2\sigma^0\bar{\lambda}\bar{\psi}_{\dot{\alpha}}
\end{aligned} \tag{E.13}$$

First line:

$$-\frac{1}{2}(\mathcal{F}'' - \mathcal{F}^{\dagger''})\hat{v}^{0i}\epsilon_2^\alpha\sigma_{i\alpha\dot{\alpha}} = -i\mathcal{I}(\epsilon_2\sigma^{\mu\nu}\sigma^0)_{\dot{\alpha}}v_{\mu\nu} \tag{E.14}$$

Second line:

$$\frac{1}{2\sqrt{2}}\mathcal{F}'''(\epsilon_2\lambda\psi^\alpha\sigma^0_{\alpha\dot{\alpha}} + \epsilon_2\psi\lambda^\alpha\sigma^0_{\alpha\dot{\alpha}}) = -\frac{1}{2\sqrt{2}}\mathcal{F}'''\psi\lambda\epsilon_2^\alpha\sigma^0_{\alpha\dot{\alpha}} \tag{E.15}$$

Third line:

$$\frac{1}{2\sqrt{2}}\mathcal{F}^{\dagger'''}(\epsilon_2\sigma^0\bar{\psi}\bar{\lambda}_{\dot{\alpha}} + \epsilon_2\sigma^0\bar{\lambda}\bar{\psi}_{\dot{\alpha}}) = -\frac{1}{2\sqrt{2}}\mathcal{F}^{\dagger'''}\bar{\psi}\bar{\lambda}\epsilon_2^\alpha\sigma^0_{\alpha\dot{\alpha}} \tag{E.16}$$

Collecting the terms in (E.14), (E.15) and (E.16) we have

$$i\mathcal{I}\sigma^0_{\alpha\dot{\alpha}}\Delta_2\psi^\alpha = -i\mathcal{I}(\epsilon_2\sigma^{\mu\nu}\sigma^0)_{\dot{\alpha}}v_{\mu\nu} - \frac{1}{2\sqrt{2}}\mathcal{F}'''\psi\lambda\epsilon_2^\alpha\sigma^0_{\alpha\dot{\alpha}} - \frac{1}{2\sqrt{2}}\mathcal{F}^{\dagger'''}\bar{\psi}\bar{\lambda}\epsilon_2^\alpha\sigma^0_{\alpha\dot{\alpha}} \tag{E.17}$$

or

$$\Delta_2\psi^\beta = -\epsilon_2^\alpha(\sigma^{\mu\nu})_\alpha^\beta v_{\mu\nu} - i\epsilon_2^\beta(-\frac{1}{2\sqrt{2}}(f\psi\lambda + f^\dagger\bar{\psi}\bar{\lambda})) = \delta_2^{\mathrm{on}}\psi^\beta \tag{E.18}$$

E.3 The transformations from Q_1^{II}

The charge is given by

$$\epsilon_1 Q_1^{II} = \int d^3x\Big(\sqrt{2}\mathcal{I}\epsilon_1(\partial\!\!\!/ A^\dagger)\bar{\sigma}^0\psi + \Pi^i\delta_1 v_i - \frac{1}{2i}\mathcal{F}''^\dagger\epsilon_1\sigma_i\bar{\lambda}v^{*0i} - \frac{1}{2\sqrt{2}}\mathcal{F}'''^\dagger\epsilon_1\sigma^0\bar{\psi}\bar{\lambda}^2\Big) \tag{E.19}$$

let us write again the momenta

$$\pi_A^{II} = -I\partial^0 A^\dagger + \frac{1}{2}\mathcal{F}'''(\psi\sigma^0\bar{\psi} + \lambda\sigma^0\bar{\lambda}) \quad \pi_{A^\dagger}^{II} = \pi_{A^\dagger} \quad \Pi^{IIi} = \Pi^i \tag{E.20}$$

$$(\pi_{\bar{\psi}}^{II})_{\dot{\alpha}} = 0 \quad (\pi_\psi^{II})^\alpha = i\mathcal{I}\bar{\psi}_{\dot{\alpha}}\bar{\sigma}^{0\dot{\alpha}\alpha} \quad (\pi_{\bar{\lambda}}^{II})_{\dot{\alpha}} = 0 \quad (\pi_\lambda^{II})^\alpha = i\mathcal{I}\bar{\lambda}_{\dot{\alpha}}\bar{\sigma}^{0\dot{\alpha}\alpha} \tag{E.21}$$

The charge re-expressed

$$\begin{aligned}\epsilon_1 Q_1^{II} &= \int d^3x\Big(\sqrt{2}\epsilon_1\psi\pi_A^{II} - \frac{1}{\sqrt{2}}\mathcal{F}'''\epsilon_1\psi(\psi\sigma^0\bar{\psi} + \lambda\sigma^0\bar{\lambda}) + \sqrt{2}\mathcal{I}\epsilon_1\sigma^i\bar{\sigma}^0\psi\partial_i A^\dagger \\ &+ \Pi^i\delta_1 v_i - \frac{1}{2i}\mathcal{F}''^\dagger\epsilon_1\sigma_i\bar{\lambda}v^{*0i} - \frac{1}{2\sqrt{2}}\mathcal{F}'''^\dagger\epsilon_1\sigma^0\bar{\psi}\bar{\lambda}^2\Big)\end{aligned} \tag{E.22}$$

where the first line in (E.22) corresponds to the first term in (E.19). Let us call Δ_1 the transformations induced by this charge.

Bosonic transformations.

When we commute the charge (E.22) with A according to the canonical Poisson brackets we only have contribution from the first term therefore $\Delta_1 A = \delta_1 A$. Trivially we see that $\Delta_1 A^\dagger = 0 = \delta_1 A^\dagger$ and $\Delta_1 v_i = \delta_1 v_i$.

Fermionic transformations.

The interesting part is the commutation of the fermions. Let us start with $\Delta_1\psi$. First we commute the last term in (E.22) that can be written as

$$-\frac{1}{2\sqrt{2}}\mathcal{F}'''^{\dagger}\epsilon_1\sigma^0\bar{\psi}\bar{\lambda}^2 = \frac{i}{2\sqrt{2}}f^\dagger\bar{\lambda}^2\epsilon_1\pi_\psi^{II} \tag{E.23}$$

and we see immediately that it is not enough to produce the expression (3.73) of $E_{\rm on}$, therefore we need also the piece introduced in the first line to write the canonical momentum for A. The relevant term there is

$$-\frac{1}{\sqrt{2}}\mathcal{F}'''\epsilon_1\psi\psi\sigma^0\bar{\psi} = -\frac{i}{2\sqrt{2}}f\psi^2\epsilon_1\pi_\psi^{II} \tag{E.24}$$

Now it is clear that

$$\begin{aligned}\Delta_1\psi_\alpha &\equiv \{\epsilon_1 Q_1^{II}\ ,\ \psi_\alpha\}_- \\ &= \sqrt{2}\epsilon_{1\alpha}(\frac{i}{4}(f^\dagger\bar{\lambda}^2 - f\psi^2)) = \sqrt{2}\epsilon_{1\alpha}E_{\rm on} = \delta_1\psi_\alpha \end{aligned}\tag{E.25}$$

Similarly for $\Delta_1\lambda$ when we consider the terms in the second line of (E.22) they are not enough to give the right expression of $D_{\rm on}$ in (3.71) and also the term $-\frac{1}{\sqrt{2}}\mathcal{F}'''\epsilon_1\psi\lambda\sigma^0\bar{\lambda}$ in the first line has to be considered. We do not show the explicit computation being in any respect identical to the one we have done with $\epsilon_1 Q^I$. At this end one could use the independent Poisson (3.104) in both cases.

Let us show in some details what happens for the other two fermions $\bar{\lambda}$ and $\bar{\psi}$. The re-expressed charge (E.22) has the term $-\frac{1}{\sqrt{2}}\mathcal{F}'''\epsilon_1\psi\lambda\sigma^0\bar{\lambda}$, therefore

we have to commute it

$$\Delta_1 \pi_\lambda^{\alpha II} \equiv \{\epsilon_1 Q_1^{II}\ ,\ \pi_\lambda^{\alpha II}\}_- = \frac{1}{\sqrt{2}} \mathcal{F}''' \epsilon_1 \psi (\bar{\lambda} \bar{\sigma}^0)^\alpha \tag{E.26}$$

and

$$\Delta_1 \pi_\lambda^{II\alpha} = \frac{1}{\sqrt{2}} \mathcal{F}''' \epsilon_1 \psi (\bar{\lambda} \bar{\sigma}^0)^\alpha + i \mathcal{I} \bar{\sigma}^{0\dot{\alpha}\alpha} \Delta_1 \bar{\lambda}_{\dot{\alpha}} \tag{E.27}$$

equating the two expressions we have the wanted $\Delta_1 \bar{\lambda}_{\dot{\alpha}} = 0 = \delta_1 \bar{\lambda}_{\dot{\alpha}}$.

Finally $\Delta_1 \bar{\psi}$. The only contributions come from the first line of (E.22)

$$\begin{aligned} \Delta_1 \pi_\psi^{\alpha II} &\equiv \{\epsilon_1 Q_1^{II}\ ,\ \pi_\psi^{\alpha II}\}_- \\ &= \sqrt{2} \pi_A^{II} \epsilon_1^\alpha + \frac{1}{\sqrt{2}} \mathcal{F}''' \epsilon_1 \sigma^0 \bar{\psi} \psi^\alpha - \frac{1}{\sqrt{2}} \mathcal{F}''' \lambda \sigma^0 \bar{\lambda} \epsilon_1^\alpha + \sqrt{2} \mathcal{I} (\epsilon_1 \sigma^i \bar{\sigma}^0)^\alpha \partial_i A^\dagger \end{aligned} \tag{E.28}$$

using the Fierz identity $\psi^\alpha \epsilon_1^\beta = \psi^\beta \epsilon_1^\alpha - \epsilon^{\alpha\beta} \epsilon_1 \psi$ we have

$$\begin{aligned} \Delta_1 \pi_\psi^{\alpha II} &= \sqrt{2} \epsilon_1^\alpha (\pi_A^{II} - \frac{1}{2} \mathcal{F}''' (\psi \sigma^0 \bar{\psi} + \lambda \sigma^0 \bar{\lambda})) + \sqrt{2} \mathcal{I} (\epsilon_1 \sigma^i \bar{\sigma}^0)^\alpha \partial_i A^\dagger \\ &\quad + \frac{1}{\sqrt{2}} \mathcal{F}''' \epsilon_1 \psi (\bar{\psi} \bar{\sigma}^0)^\alpha \\ &= \sqrt{2} \mathcal{I} (\epsilon_1 \sigma^\mu \bar{\sigma}^0)^\alpha \partial_\mu A^\dagger + \frac{1}{\sqrt{2}} \mathcal{F}''' \epsilon_1 \psi (\bar{\psi} \bar{\sigma}^0)^\alpha \end{aligned} \tag{E.29}$$

which combined with the usual

$$\Delta_1 \pi_\psi^{\alpha II} = \frac{1}{\sqrt{2}} \mathcal{F}''' \epsilon_1 \psi (\bar{\psi} \bar{\sigma}^0)^\alpha + i \mathcal{I} \bar{\sigma}^{0\dot{\alpha}\alpha} \Delta_1 \bar{\psi}_{\dot{\alpha}} \tag{E.30}$$

gives the wanted

$$\Delta_1 \bar{\psi}_{\dot{\alpha}} = -i \epsilon_1^\alpha\ \partial\!\!\!/_{\alpha\dot{\alpha}} A^\dagger = \delta_1 \bar{\psi}_{\dot{\alpha}} \tag{E.31}$$

Thus we conclude that $\Delta_1 \equiv \delta_1$ also in the $\mathcal{L}^{II}$-*setting* therefore this is a final proof that the canonical procedure works even if some labour is needed. Note that we could not get the right transformations for the spinors if we had used the charge in (E.19).

E.4 Transformations of the dummy fields

We want to show here that the transformations of the dummy fields on-shell can be obtained by the transformations of the fermions. At this end let us write again the Euler-Lagrange equations for $E, E^\dagger$ and D

$$D = -\frac{1}{2\sqrt{2}}(f\psi\lambda + f^\dagger\bar{\psi}\bar{\lambda}) \tag{E.32}$$

$$E^\dagger = -\frac{i}{4}(f\lambda^2 - f^\dagger\bar{\psi}^2) \tag{E.33}$$

$$E = \frac{i}{4}(f^\dagger\bar{\lambda}^2 - f\psi^2) \tag{E.34}$$

The Euler-Lagrange equations for the fermions, obtained from the Lagrangian (3.66), are given by

$$\bar{\partial\!\!\!/}^{\dot{\alpha}\alpha}\psi_\alpha = \frac{i}{2}f(\bar{\partial\!\!\!/}^{\dot{\alpha}\alpha}A)\psi_\alpha - \frac{1}{2}f^\dagger(\frac{1}{\sqrt{2}}\bar{\sigma}^{\mu\nu\dot{\alpha}}{}_{\dot{\beta}}\bar{\lambda}^{\dot{\beta}}v_{\mu\nu} + E\bar{\psi}^{\dot{\alpha}} + \frac{i}{\sqrt{2}}D\bar{\lambda}^{\dot{\alpha}}) + \frac{1}{4}g^\dagger\bar{\psi}^{\dot{\alpha}}\bar{\lambda}\bar{\lambda} \tag{E.35}$$

$$\partial\!\!\!/_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = -\frac{i}{2}f^\dagger(\partial\!\!\!/_{\alpha\dot{\alpha}}A^\dagger)\bar{\psi}^{\dot{\alpha}} + \frac{1}{2}f(\frac{1}{\sqrt{2}}\sigma^{\mu\nu\beta}_{\alpha}\lambda_\beta v_{\mu\nu} + E^\dagger\psi_\alpha - \frac{i}{\sqrt{2}}D\lambda_\alpha) - \frac{1}{4}g\psi_\alpha\lambda\lambda \tag{E.36}$$

$$\bar{\partial\!\!\!/}^{\dot{\alpha}\alpha}\lambda_\alpha = \frac{i}{2}f(\bar{\partial\!\!\!/}^{\dot{\alpha}\alpha}A)\lambda_\alpha - \frac{1}{2}f^\dagger(-\frac{1}{\sqrt{2}}\bar{\sigma}^{\mu\nu\dot{\alpha}}{}_{\dot{\beta}}\bar{\psi}^{\dot{\beta}}v_{\mu\nu} + E^\dagger\bar{\lambda}^{\dot{\alpha}} + \frac{i}{\sqrt{2}}D\bar{\psi}^{\dot{\alpha}}) + \frac{1}{4}g^\dagger\bar{\lambda}^{\dot{\alpha}}\bar{\psi}\bar{\psi} \tag{E.37}$$

$$\partial\!\!\!/_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} = -\frac{i}{2}f^\dagger(\partial\!\!\!/_{\alpha\dot{\alpha}}A^\dagger)\bar{\lambda}^{\dot{\alpha}} + \frac{1}{2}f(-\frac{1}{\sqrt{2}}\sigma^{\mu\nu\beta}_{\alpha}\psi_\beta v_{\mu\nu} + E\lambda_\alpha - \frac{i}{\sqrt{2}}D\psi_\alpha) - \frac{1}{4}g\lambda_\alpha\psi\psi \tag{E.38}$$

where $f(A, A^\dagger) \equiv \mathcal{F}'''/\mathcal{I}$ and $g(A, A^\dagger) \equiv \mathcal{F}''''/\mathcal{I}$. Note that after integration by parts nothing happens to (E.32), (E.33) and (E.34), whereas, of course, some of the Euler-Lagrange equations for the fermions become meaningless. After a lengthy computation we obtain

$$\delta_1 E = 0 \tag{E.39}$$

$$\begin{aligned} -\frac{i}{\sqrt{2}}\delta_1 E^\dagger &= \epsilon_1^\alpha\Big[- f^\dagger(\frac{i}{2}(\partial\!\!\!/_{\alpha\dot{\alpha}}A^\dagger)\bar{\psi}^{\dot{\alpha}}) + \frac{1}{2}\lambda_\alpha[(g - \frac{1}{2i}f^2)\psi\lambda - \frac{1}{i4\sqrt{2}}ff^\dagger\bar{\psi}\bar{\lambda}] \\ &+ \frac{1}{2\sqrt{2}}f(\sqrt{2}\psi_\alpha E^\dagger + (\sigma^{\mu\nu}\lambda)_\alpha v_{\mu\nu} + i\lambda_\alpha D)\Big] \end{aligned} \tag{E.40}$$

and

$$\begin{aligned} -2\sqrt{2}\delta_1 D &= \epsilon_1^\alpha \Big[-f^\dagger(i\sqrt{2}(\partial\!\!\!/_{\alpha\dot\alpha} A^\dagger)\bar\lambda^{\dot\alpha}) + \sqrt{2}\psi_\alpha[(g - \frac{1}{2i}f^2)\psi\lambda - \frac{1}{2i}ff^\dagger\bar\psi\bar\lambda] \\ &+ f(\sqrt{2}\lambda_\alpha E - (\sigma^{\mu\nu}\psi)_\alpha v_{\mu\nu} + i\psi_\alpha D)\Big] \end{aligned} \tag{E.41}$$

where we used

$$\delta f = \delta A(g - \frac{1}{2i}f^2) + \delta A^\dagger \frac{1}{2i}ff^\dagger \tag{E.42}$$

Comparing these expressions with the Euler-Lagrange equations for the fermions we have

$$\delta_1 E = 0 \quad \delta_1 E^\dagger = i\sqrt{2}\epsilon_1\ \partial\!\!\!/\bar\psi \quad \delta_1 D = -\epsilon_1\ \partial\!\!\!/\bar\lambda \tag{E.43}$$

in agreement with the given Susy variations.

Appendix F

The SU(2) computations

In this Appendix we collect all the formulae and computations relevant for our analysis of the SW SU(2) effective theory.

F.1 Properties of $\mathcal{F}^{a_1\cdots a_n}$

Some care is necessary in handling the derivatives of the prepotential $\mathcal{F}(A^aA^a)$, function of the SU(2) Casimir A^aA^a. The first four derivatives are given by

$$\mathcal{F}^a = 2A^a\mathcal{F}' \tag{F.1}$$

$$\mathcal{F}^{ab} = 2\delta^{ab}\mathcal{F}' + 4A^aA^b\mathcal{F}'' \tag{F.2}$$

$$\mathcal{F}^{abc} = 4(\delta^{ab}A^c + \delta^{ac}A^b + \delta^{bc}A^a)\mathcal{F}'' + 8A^aA^bA^c\mathcal{F}''' \tag{F.3}$$

$$\begin{aligned}\mathcal{F}^{abcd} = {} & 4\mathcal{F}''(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{bc}\delta^{ad}) + 8\mathcal{F}'''(A^aA^b\delta^{cd} + A^aA^c\delta^{bd} + A^aA^d\delta^{bc} \\ & + A^bA^c\delta^{ad} + A^bA^d\delta^{ac} + A^cA^d\delta^{ab}) + 16\mathcal{F}''''A^aA^bA^cA^d\end{aligned} \tag{F.4}$$

similarly for $\mathcal{F}^\dagger$.

Form the expressions (F.1)-(F.4) it is easy to obtain the following very useful identities:

$$\epsilon^{abc}\mathcal{F}^{bd}A^c = \epsilon^{adc}\mathcal{F}^c \tag{F.5}$$

$$\epsilon^{bcd}\mathcal{F}^{be}A^{d} = -\epsilon^{bed}\mathcal{F}^{bc}A^{d} \tag{F.6}$$

$$\mathcal{F}^{abc}\epsilon^{cde}A^{e} = \mathcal{F}^{be}\epsilon^{ade} + \mathcal{F}^{ae}\epsilon^{bde} \tag{F.7}$$

similarly for $\mathcal{F}^{\dagger}$.

Properties (F.5)-(F.7) are extensively used throughout the SU(2) computations. As an important example we want to show the explicit computation of the bosonic coefficients of the spinor terms entering the Gauss constraint in the expression (4.49) of the central charge. These terms are given by

$$i\sqrt{2}[i\mathcal{I}^{be}\epsilon^{bcd}A^{\dagger d}+\frac{i}{4}\mathcal{I}^{be}\epsilon^{bcd}A^{\dagger d}+\frac{i}{4}\mathcal{I}^{bc}\epsilon^{bed}A^{\dagger d}+\frac{1}{8}\mathcal{F}^{ace}\epsilon^{adb}A^{d}A^{\dagger b}](\psi^{e}\sigma^{0}\bar{\psi}^{c}+\lambda^{e}\sigma^{0}\bar{\lambda}^{c}) \tag{F.8}$$

By expanding the terms in square brackets we obtain:

$$\begin{aligned}
[\frac{1}{2}\mathcal{F}^{be}\epsilon^{bcd}A^{\dagger d} &- \frac{1}{2}\mathcal{F}^{\dagger be}\epsilon^{bcd}A^{\dagger d} \\
+\frac{1}{8}\mathcal{F}^{be}\epsilon^{bcd}A^{\dagger d} &- \frac{1}{8}\mathcal{F}^{\dagger be}\epsilon^{bcd}A^{\dagger d} \\
+\frac{1}{8}\mathcal{F}^{bc}\epsilon^{bed}A^{\dagger d} &- \frac{1}{8}\mathcal{F}^{\dagger bc}\epsilon^{bed}A^{\dagger d} \\
-\frac{1}{8}\mathcal{F}^{cd}\epsilon^{ebd}A^{\dagger b} &- \frac{1}{8}\mathcal{F}^{ed}\epsilon^{cbd}A^{\dagger b}]
\end{aligned} \tag{F.9}$$

where the identity (F.7) was used to write the last term. By collecting similar terms and using the property (F.6) we end up with

$$[\frac{1}{2}\mathcal{F}^{be}\epsilon^{bcd}A^{\dagger d} - \frac{1}{2}\mathcal{F}^{\dagger be}\epsilon^{bcd}A^{\dagger d}] = i\mathcal{I}^{be}\epsilon^{bcd}A^{\dagger d} \tag{F.10}$$

which is the correct coefficient according to the Gauss law (4.34).

F.2 Computation of the Hamiltonian

First we have to conveniently write $\bar{\epsilon}_1\bar{Q}_1$ in (4.21) introducing $\pi_{A^\dagger}$ (see the discussion at the end of Section 3.3.3 and Appendix E). The Poisson

brackets are then given by

$$\begin{aligned}\{\epsilon_1 Q_1, \bar{\epsilon}_1\bar{Q}_1\}_- \;=\; & \int d^3x d^3y \{\Pi^{ai}\delta_1 v_i^a + \delta_1\bar{\psi}^a\pi^a_{\bar{\psi}} + \frac{i}{2}\mathcal{F}^{\dagger ab}\epsilon_1\sigma_i\bar{\lambda}^a v^{*0ib} \\ & -\frac{1}{2\sqrt{2}}\mathcal{F}^{\dagger abc}\epsilon_1\sigma^0\bar{\psi}^a\bar{\lambda}^b\bar{\lambda}^c + i\mathcal{I}^{ab}\epsilon_1\sigma^0\bar{\lambda}^b\epsilon^{acd}A^c A^{d\dagger}\ , \\ & \Pi^{ej}\bar{\delta}_1 v_j^e + \bar{\delta}_1 A^{e\dagger}\pi^e_{A^\dagger} + \frac{1}{\sqrt{2}}\bar{\epsilon}_1\bar{\psi}^e\mathcal{F}^{efg\dagger}(\bar{\psi}^f\bar{\sigma}^0\psi^g + \bar{\lambda}^f\bar{\sigma}^0\lambda^g) \\ & +\sqrt{2}\mathcal{I}^{ef}\bar{\epsilon}_1\bar{\sigma}^j\sigma^0\bar{\psi}^f\mathcal{D}_j A^e + \frac{i}{2}\mathcal{F}^{ef}\bar{\epsilon}_1\bar{\sigma}_j\lambda^e v^{*0jf} \\ & +\frac{1}{2\sqrt{2}}\mathcal{F}^{efg}\bar{\epsilon}_1\bar{\sigma}^0\psi^e\lambda^f\lambda^g - \bar{\epsilon}_1\pi^e_{\bar{\lambda}}\epsilon^{efg}A^f A^{g\dagger}\}_- \end{aligned} \tag{F.11}$$

where $\delta_1 v_i^a = i\epsilon_1\sigma_i\bar{\lambda}^a$, $\bar{\delta}_1 v_j^e = i\bar{\epsilon}_1\bar{\sigma}_j\lambda^e$, $\bar{\delta}_1 A^{e\dagger} = \sqrt{2}\bar{\epsilon}_1\bar{\psi}^e$, $\delta_1 A^b = \sqrt{2}\epsilon_1\psi^b$, $\delta_1\bar{\psi}^a = -i\sqrt{2}\epsilon_1\ \not{\mathcal{D}}A^{a\dagger}$,and $\delta_1\bar{\psi}^a\pi^a_{\bar{\psi}} = \delta_1 A^b\pi^b_A + \sqrt{2}\mathcal{I}^{ab}\epsilon_1\sigma^i\bar{\sigma}^0\psi^b\mathcal{D}_i A^{a\dagger}$. Let us write

$$\{\epsilon_1 Q_1, \bar{\epsilon}_1\bar{Q}_1\}_- = \int d^3x d^3y \Big(\mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V} + \mathrm{VI}\Big) \tag{F.12}$$

where

$$\begin{aligned}\text{terms I} \;=\; & -\Pi^{ai}\Pi^{ej}\{\epsilon_1\sigma_i\bar{\lambda}^a\ ,\ \bar{\epsilon}_1\bar{\sigma}_j\lambda^e\}_- \\ & +\frac{i}{\sqrt{2}}\mathcal{F}^{efg\dagger}\Pi^{ai}\bar{\epsilon}_i\bar{\psi}^e\{\epsilon_1\sigma_i\bar{\lambda}^a\ ,\ \bar{\lambda}^f\bar{\sigma}^0\lambda^g\}_- \\ & +i\sqrt{2}\mathcal{I}^{ef}\epsilon_1\sigma_i\bar{\lambda}^a\bar{\epsilon}_1\bar{\sigma}^j\sigma^0\psi^f\{\Pi^{ai}\ ,\ \mathcal{D}_j A^e\ \}_- \\ & -\frac{1}{2}\mathcal{F}^{ef}\epsilon_1\sigma_i\bar{\lambda}^a\bar{\epsilon}_1\bar{\sigma}_j\lambda^e\{\Pi^{ai}\ ,\ v^{*0jf}\}_- \\ & -\frac{1}{2}\mathcal{F}^{ef}\{\epsilon_1\sigma_i\bar{\lambda}^a\ ,\ \bar{\epsilon}_1\bar{\sigma}_j\lambda^e\}_-\Pi^{ai}v^{*0jf} \\ & +\frac{i}{2\sqrt{2}}\mathcal{F}^{efg}\bar{\epsilon}_1\bar{\sigma}^0\psi^e\Pi^{ai}\{\epsilon_1\sigma_i\bar{\lambda}^a\ ,\ \lambda^f\lambda^g\}_- \\ & -i\epsilon^{efg}A^f A^{\dagger g}\Pi^{ai}\{\epsilon_1\sigma_i\bar{\lambda}^a\ ,\ \bar{\epsilon}_1\pi^e_{\bar{\lambda}}\}_- \end{aligned} \tag{F.13}$$

$$\begin{aligned}\text{terms II} \;=\; & +i\sqrt{2}\mathcal{I}^{ab}\epsilon_1\sigma^\mu\bar{\sigma}^0\psi^b\bar{\epsilon}_1\bar{\sigma}_j\lambda^e\{\mathcal{D}_\mu A^{\dagger a}\ , \Pi^{ej}\ \}_- \\ & +2\mathcal{I}^{ab}\epsilon_1\sigma^i\bar{\sigma}^0\psi^b\bar{\epsilon}_1\bar{\psi}^e\{\mathcal{D}_i A^{\dagger a}\ , \pi^e_{A^\dagger}\ \}_- \\ & -i2(\epsilon_1\ \not{\mathcal{D}}A^{\dagger a})_{\dot{\alpha}}\{\pi^{a\dot{\alpha}}_{\bar{\psi}}\ ,\ \bar{\epsilon}_1\bar{\psi}^e\}_-\pi^e_{A^\dagger} \end{aligned}$$

$$
\begin{aligned}
&-i(\epsilon_1 \not{\mathcal{D}} A^{\dagger a})_{\dot\alpha}\{\pi_{\bar\psi}^{a\dot\alpha}\ ,\ \bar\epsilon_1\bar\psi^e\bar\psi^f\bar\sigma^0\psi^g\}_-\mathcal{F}^{efg\dagger}\\
&-i(\epsilon_1 \not{\mathcal{D}} A^{\dagger a})_{\dot\alpha}\{\pi_{\bar\psi}^{a\dot\alpha}\ ,\ \bar\epsilon_1\bar\psi^e\}_-\bar\lambda^f\bar\sigma^0\lambda^g\mathcal{F}^{efg\dagger}\\
&+2\epsilon_1\psi^a\{\pi_A^a\ ,\ \mathcal{I}^{ef}\}_-\bar\epsilon_1\bar\sigma^j\sigma^0\bar\psi^f\mathcal{D}_jA^e\\
&-i2(\epsilon_1 \not{\mathcal{D}} A^{\dagger a})_{\dot\alpha}\{\pi_{\bar\psi}^{a\dot\alpha}\ ,\ \bar\epsilon_1\bar\sigma^j\sigma^0\bar\psi^f\}_-\mathcal{I}^{ef}\mathcal{D}_jA^e\\
&+2\epsilon_1\psi^a\mathcal{I}^{ef}\bar\epsilon_1\bar\sigma^j\sigma^0\bar\psi^f\{\pi_A^a\ ,\ \mathcal{D}_jA^e\}_-\\
&+\frac{i}{\sqrt2}\epsilon_1\psi^a\{\pi_A^a\ ,\ \mathcal{F}^{ef}\}_-\bar\epsilon_1\bar\sigma_j\lambda^e v^{*0jf}\\
&+\frac{1}{2}\epsilon_1\psi^a\{\pi_A^a\ ,\ \mathcal{F}^{efg}\}_-\bar\epsilon_1\bar\sigma^0\psi^e\lambda^f\lambda^g\\
&-\sqrt2\epsilon^{efg}\bar\epsilon_1\pi_{\bar\lambda}^e A^{\dagger g}\{\pi_A^a\ ,\ A^f\}_-\epsilon_1\psi^a
\end{aligned}
\qquad\text{(F.14)}
$$

$$
\begin{aligned}
\text{terms III}\quad=\quad&-\frac{1}{2}\mathcal{F}^{\dagger ab}\Pi^{ej}v^{*oib}\{\epsilon_1\sigma_i\bar\lambda^a\ ,\ \bar\epsilon_1\bar\sigma_j\lambda^e\}_-\\
&-\frac{1}{2}\mathcal{F}^{\dagger ab}\{v^{*oib}\ ,\ \Pi^{ej}\}_-\epsilon_1\sigma_i\bar\lambda^a\bar\epsilon_1\bar\sigma_j\lambda^e\\
&+\frac{i}{\sqrt2}\bar\epsilon_1\bar\psi^e\{\mathcal{F}^{ab\dagger}\ ,\ \pi_{A^\dagger}^e\}_-\epsilon_1\sigma_i\bar\lambda^a v^{*0ib}\\
&+\frac{i}{2\sqrt2}\mathcal{F}^{ab\dagger}\bar\epsilon_1\bar\psi^e\{\epsilon_1\sigma_i\bar\lambda^a\ ,\ \bar\lambda^f\bar\sigma^0\lambda^g\}_-v^{*0ib}\mathcal{F}^{efg\dagger}\\
&-\frac{1}{4}\mathcal{F}^{\dagger ab}\mathcal{F}^{ef}v^{*0ib}v^{*0jf}\{\epsilon_1\sigma_i\bar\lambda^a\ ,\ \bar\epsilon_1\bar\sigma_j\lambda^e\}_-\\
&+\frac{i}{4\sqrt2}\mathcal{F}^{\dagger ab}\mathcal{F}^{efg}v^{*0ib}\bar\epsilon_1\bar\sigma^0\psi^e\{\epsilon_1\sigma_i\bar\lambda^a\ ,\ \lambda^f\lambda^g\}_-\\
&+\frac{i}{2}\mathcal{F}^{\dagger ab}\epsilon^{efg}A^fA^{\dagger g}v^{*0ib}\bar\epsilon_{1\dot\alpha}\{\pi_{\bar\lambda}^{e\dot\alpha}\ ,\ \epsilon_1\sigma_i\bar\lambda^a\}_-
\end{aligned}
\qquad\text{(F.15)}
$$

$$
\begin{aligned}
\text{terms IV}\quad=\quad&-\frac{i}{2\sqrt2}\mathcal{F}^{\dagger abc}\Pi^{ej}\epsilon_1\sigma^0\bar\psi^a\{\bar\lambda^b\bar\lambda^c\ ,\ \bar\epsilon_1\bar\sigma_j\lambda^e\}_-\\
&-\frac{1}{2}\epsilon_1\sigma^0\bar\psi^a\bar\lambda^b\bar\lambda^c\bar\epsilon_1\bar\psi^e\{\mathcal{F}^{\dagger abc}\ ,\ \pi_{A^\dagger}^e\}_-\\
&-\frac{1}{4}\mathcal{F}^{\dagger abc}\mathcal{F}^{\dagger efg}\bar\epsilon_1\bar\psi^e\bar\lambda^b\bar\lambda^c\{\epsilon_1\sigma^0\bar\psi^a\ ,\ \bar\psi^f\bar\sigma^0\psi^g\}_-\\
&-\frac{1}{4}\mathcal{F}^{\dagger abc}\mathcal{F}^{\dagger efg}\bar\epsilon_1\bar\psi^e\epsilon_1\sigma^0\bar\psi^a\{\bar\lambda^b\bar\lambda^c\ ,\ \bar\lambda^f\bar\sigma^0\lambda^g\}_-\\
&-\frac{i}{4\sqrt2}\mathcal{F}^{\dagger abc}\mathcal{F}^{ef}v^{*0jf}\epsilon_1\sigma^0\bar\psi^a\{\bar\lambda^b\bar\lambda^c\ ,\ \bar\epsilon_1\bar\sigma_j\lambda^e\}_-
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{8}\mathcal{F}^{\dagger abc}\mathcal{F}^{efg}\bar{\lambda}^b\bar{\lambda}^c\lambda^f\lambda^g\{\epsilon_1\sigma^0\bar{\psi}^a\ ,\ \bar{\epsilon}_1\bar{\sigma}^0\psi^e\}_- \\
&-\frac{1}{8}\mathcal{F}^{\dagger abc}\mathcal{F}^{efg}\{\bar{\lambda}^b\bar{\lambda}^c\ ,\ \lambda^f\lambda^g\}_-\epsilon_1\sigma^0\bar{\psi}^a\bar{\epsilon}_1\bar{\sigma}^0\psi^e \\
&-\frac{1}{2\sqrt{2}}\mathcal{F}^{\dagger abc}\epsilon_1\sigma^0\bar{\psi}^a\bar{\epsilon}_{1\dot{\alpha}}\{\pi^{e\dot{\alpha}}_{\bar{\lambda}}\ ,\ \bar{\lambda}^b\bar{\lambda}^c\}_-\epsilon^{efg}A^fA^{\dagger g} \qquad \text{(F.16)}
\end{aligned}
$$

$$
\begin{aligned}
\text{terms V} \quad = \quad &-\epsilon^{acd}\mathcal{I}^{ab}\{\epsilon_1\sigma^0\bar{\lambda}^b\ ,\ \bar{\epsilon}_1\bar{\sigma}_j\lambda^e\}_-\Pi^{ej}A^cA^{\dagger d} \\
&+i\sqrt{2}\epsilon^{acd}\{\mathcal{I}^{ab}\ ,\ \pi^e_{A^\dagger}\}_-\bar{\epsilon}_1\bar{\psi}^e\epsilon_1\sigma^0\bar{\lambda}^bA^cA^{\dagger d} \\
&+i\sqrt{2}\epsilon^{acd}\epsilon_1\sigma^0\bar{\lambda}^b\bar{\epsilon}_1\bar{\psi}^eA^c\{A^{\dagger d}\ ,\ \pi^e_{A^\dagger}\}_-\mathcal{I}^{ab} \\
&+\frac{i}{\sqrt{2}}\epsilon^{acd}\mathcal{I}^{ab}\mathcal{F}^{\dagger efg}A^cA^{\dagger d}\bar{\epsilon}_1\bar{\psi}^e\{\epsilon_1\sigma^0\bar{\lambda}^b\ ,\ \bar{\lambda}^f\bar{\sigma}^0\lambda^g\}_- \\
&-\frac{1}{2}\epsilon^{acd}\mathcal{I}^{ab}\mathcal{F}^{ef}A^cA^{\dagger d}v^{*0jf}\{\epsilon_1\sigma^0\bar{\lambda}^b\ ,\ \bar{\epsilon}_1\bar{\sigma}_j\lambda^e\}_- \\
&+\frac{i}{2\sqrt{2}}\epsilon^{acd}\mathcal{I}^{ab}\mathcal{F}^{efg}A^cA^{\dagger d}\bar{\epsilon}_1\bar{\sigma}^0\psi^e\{\epsilon_1\sigma^0\bar{\lambda}^b\ ,\ \lambda^f\lambda^g\}_- \\
&-i\epsilon^{acd}\epsilon^{efg}\mathcal{I}^{ab}A^cA^{\dagger d}A^fA^{\dagger g}\{\epsilon_1\sigma^0\bar{\lambda}^b\ ,\ \bar{\epsilon}_1\pi^e_{\bar{\lambda}}\}_- \qquad \text{(F.17)}
\end{aligned}
$$

$$
\begin{aligned}
\text{terms VI} \quad = \quad &+i\sqrt{2}\epsilon_1\psi^a\Pi^{ej}\{\pi^a_A\ ,\ \bar{\epsilon}_1\bar{\sigma}_j\lambda^e\}_- \\
&+\bar{\epsilon}_1\bar{\psi}^e\epsilon_1\psi^a\mathcal{F}^{\dagger efg}[(\bar{\psi}^f\bar{\sigma}^0)^\alpha\{\pi^a_A\ ,\ \psi^g_\alpha\}_- + (\bar{\lambda}^f\bar{\sigma}^0)^\alpha\{\pi^a_A\ ,\ \lambda^g_\alpha\}_-] \\
&+\frac{i}{\sqrt{2}}\epsilon_1\psi^a\mathcal{F}^{ef}v^{*0jf}(\bar{\epsilon}_1\bar{\sigma}_j)^\alpha\{\pi^a_A\ ,\ \lambda^e_\alpha\}_- \\
&+\frac{1}{2}\epsilon_1\psi^a\mathcal{F}^{efg}[(\bar{\epsilon}_1\bar{\sigma}^0)^\alpha\{\pi^a_A\ ,\ \psi^e_\alpha\}_-\lambda^f\lambda^g + \bar{\epsilon}_1\bar{\sigma}^0\psi^e\{\pi^a_A\ ,\ \lambda^f\lambda^g\}_-] \\
&\qquad \text{(F.18)}
\end{aligned}
$$

where we kept explicitly the non trivial terms VI. It is now matter to explicitly compute the Poisson brackets. At this end let us write the following useful formulae

$$
\begin{aligned}
\{\pi^a_A\ ,\ \psi^b_\alpha\}_- \quad &= \quad -\frac{i}{2}(\mathcal{I}^{bc})^{-1}\mathcal{F}^{cad}\psi^d_\alpha \qquad \text{(F.19)} \\
\{\pi^a_{A^\dagger}\ ,\ \psi^b_\alpha\}_- \quad &= \quad +\frac{i}{2}(\mathcal{I}^{bc})^{-1}\mathcal{F}^{\dagger cad}\psi^d_\alpha \qquad \text{(F.20)}
\end{aligned}
$$

same for λ (these are responsible for terms VI)

$$
\{\bar{\psi}^a_{\dot{\alpha}}\ ,\ \psi^b_\alpha\}_+ = \{\bar{\lambda}^a_{\dot{\alpha}}\ ,\ \lambda^b_\alpha\}_+ = -i(\mathcal{I}^{ab})^{-1}\sigma^0_{\alpha\dot{\alpha}} \qquad \text{(F.21)}
$$

$$\{\epsilon_1\sigma_i\bar{\lambda}^a\ ,\ \bar{\epsilon}_1\bar{\sigma}_j\lambda^e\}_- = i(\mathcal{I}^{ae})^{-1}\epsilon_1\sigma_i\bar{\sigma}^0\sigma_j\bar{\epsilon}_1 \tag{F.22}$$

$$\{\epsilon_1\sigma_i\bar{\lambda}^a\ ,\ \bar{\lambda}^f\bar{\sigma}^0\lambda^g\}_- = i(\mathcal{I}^{ag})^{-1}\epsilon_1\sigma_i\bar{\lambda}^f \tag{F.23}$$

$$\{\Pi^{ai}\ ,\ \mathcal{D}_jA^e\}_- = \epsilon^{aeh}\delta^i_jA^h \tag{F.24}$$

$$\{\Pi^{ai}(x)\ ,\ v^{*0jf}(y)\}_- = -2\epsilon^{0ijk}(\delta^{af}\partial^y_k + \epsilon^{afh}v^h_k(y))\delta^{(3)}(\vec{x}-\vec{y}) \tag{F.25}$$

$$\{\epsilon_1\sigma_i\bar{\lambda}^a\ ,\ \lambda^f\lambda^g\}_- = -i(\mathcal{I}^{af})^{-1}\epsilon_1\sigma_i\bar{\sigma}^0\lambda^g + (f\leftrightarrow g) \tag{F.26}$$

$$\{\epsilon_1\sigma_i\bar{\lambda}^a\ ,\ \bar{\epsilon}_1\pi^e_{\bar{\lambda}}\}_- = \delta^{ae}\bar{\epsilon}_1\bar{\sigma}_i\epsilon_1 \tag{F.27}$$

$$\{\mathcal{D}_\mu A^{\dagger a}(x)\ ,\ \pi^e_{A^\dagger}(y)\}_- = (\delta^{ae}\partial^x_i + \epsilon^{ade}v^d_i(x))\delta^{(3)}(\vec{x}-\vec{y}) \tag{F.28}$$

$$\{\pi^{a\dot{\alpha}}_{\bar{\psi}}\ ,\ \bar{\epsilon}_1\bar{\psi}^e\bar{\psi}^f\bar{\sigma}^0\psi^g\}_- = \bar{\epsilon}^{\dot{\alpha}}_1\delta^{ae}\bar{\psi}^f\bar{\sigma}^0\psi^g + \bar{\epsilon}_1\bar{\psi}^e(\bar{\sigma}^0\psi^g)^{\dot{\alpha}}\delta^{af} \tag{F.29}$$

$$\{\bar{\lambda}^b\bar{\lambda}^c\ ,\ \bar{\epsilon}_1\bar{\sigma}_j\lambda^e\}_- = i(\mathcal{I}^{ec})^{-1}\bar{\epsilon}_1\bar{\sigma}_j\sigma^0\bar{\lambda}^b + (b\leftrightarrow c) \tag{F.30}$$

$$\{\epsilon_1\sigma^0\bar{\psi}^a\ ,\ \bar{\psi}^f\bar{\sigma}^0\psi^g\}_- = i(\mathcal{I}^{ag})^{-1}\epsilon_1\sigma^0\bar{\psi}^f \tag{F.31}$$

$$\{\bar{\lambda}^b\bar{\lambda}^c\ ,\ \bar{\lambda}^f\bar{\sigma}^0\lambda^g\}_- = i(\mathcal{I}^{gc})^{-1}\bar{\lambda}^f\bar{\lambda}^b + (b\leftrightarrow c) \tag{F.32}$$

$$\begin{aligned}\{\bar{\lambda}^b\bar{\lambda}^c\ ,\ \lambda^f\lambda^g\}_- &= -i(\mathcal{I}^{cf})^{-1}\bar{\lambda}^b\bar{\sigma}^0\lambda^g - i(\mathcal{I}^{bg})^{-1}\bar{\lambda}^c\bar{\sigma}^0\lambda^f \\ &\quad +(f\leftrightarrow g)\end{aligned} \tag{F.33}$$

After commutation the terms above given become

$$\begin{aligned}\text{terms I} &= -i(\mathcal{I}^{ae})^{-1}\Pi^{ai}\Pi^{ej}\epsilon_1\sigma_i\bar{\sigma}^0\sigma_j\bar{\epsilon}_1 \\ &\quad -\frac{1}{\sqrt{2}}(\mathcal{I}^{ag})^{-1}\mathcal{F}^{efg\dagger}\Pi^{ai}\bar{\epsilon}_i\bar{\psi}^e\epsilon_1\sigma_i\bar{\lambda}^f \\ &\quad -i\sqrt{2}\epsilon^{eah}\mathcal{I}^{ef}A^h\epsilon_1\sigma_i\bar{\lambda}^a\bar{\epsilon}_1\bar{\sigma}^i\sigma^0\bar{\psi}^f \\ &\quad +\int d^3xd^3y[\mathcal{F}^{ef}(y)\epsilon_1\sigma_i\bar{\lambda}^a(x)\bar{\epsilon}_1\bar{\sigma}_j\lambda^e(y)\epsilon^{0ijk} \\ &\qquad \times(\delta^{af}\partial^y_k + \epsilon^{afh}v^h_k(y))\delta^{(3)}(\vec{x}-\vec{y})] \\ &\quad -\frac{i}{2}(\mathcal{I}^{ae})^{-1}\mathcal{F}^{ef}\Pi^{ai}v^{*0jf}\epsilon_1\sigma_i\bar{\sigma}^0\sigma_j\bar{\epsilon}_1 \\ &\quad +\frac{1}{\sqrt{2}}\mathcal{F}^{efg}(\mathcal{I}^{af})^{-1}\Pi^{ai}\bar{\epsilon}_1\bar{\sigma}^0\psi^e\epsilon_1\sigma_i\bar{\sigma}^0\lambda^g \\ &\quad -i\epsilon^{efg}A^fA^{\dagger g}\Pi^{ei}\bar{\epsilon}_1\bar{\sigma}_i\epsilon_1\end{aligned} \tag{F.34}$$

$$\text{terms II} = +i\sqrt{2}\epsilon^{aeh}\mathcal{I}^{ab}\epsilon_1\sigma^i\bar{\sigma}^0\psi^b\bar{\epsilon}_1\bar{\sigma}_i\lambda^eA^{\dagger h}$$

$$
\begin{aligned}
&+\int d^3xd^3y[2\mathcal{I}^{ab}(x)\epsilon_1\sigma^i\bar{\sigma}^0\psi^b(x)\bar{\epsilon}_1\bar{\psi}^e(y) \\
&\qquad\times(\delta^{ae}\partial_i^x+\epsilon^{ade}v_i^d(x))\delta^{(3)}(\vec{x}-\vec{y})] \\
&-i2\epsilon_1\sigma^\mu\bar{\epsilon}_1\pi^a_{A^\dagger}\mathcal{D}_\mu A^{\dagger a} \\
&-i\mathcal{F}^{aeg\dagger}(\mathcal{D}_\mu A^{\dagger a})[\epsilon_1\sigma^\mu\bar{\epsilon}_1(\bar{\psi}^e\bar{\sigma}^0\psi^g+\bar{\lambda}^e\bar{\sigma}^0\lambda^g) \\
&\qquad+\epsilon_1\sigma^\mu\bar{\sigma}^0\psi^g\bar{\epsilon}_1\bar{\psi}^e] \\
&+i\epsilon_1\psi^a\mathcal{F}^{aef}\bar{\epsilon}_1\bar{\sigma}^j\sigma^0\bar{\psi}^f\mathcal{D}_jA^e \\
&-i2\epsilon_1\sigma^\mu\bar{\sigma}^0\sigma^j\bar{\epsilon}_1\mathcal{I}^{ea}(\mathcal{D}_\mu A^{\dagger e})(\mathcal{D}_jA^a) \\
&-\int d^3xd^3y[2\epsilon_1\psi^a(x)\mathcal{I}^{ef}(y)\bar{\epsilon}_1\bar{\sigma}^j\sigma^0\bar{\psi}^f(y) \\
&\qquad\times(\delta^{ae}\partial_j^y+\epsilon^{eda}v_j^d(y))\delta^{(3)}(\vec{x}-\vec{y})] \\
&-\frac{i}{\sqrt{2}}\epsilon_1\psi^a\mathcal{F}^{aef}\bar{\epsilon}_1\bar{\sigma}_j\lambda^e v^{*0jf} \\
&-\frac{1}{2}\epsilon_1\psi^a\mathcal{F}^{aefg}\bar{\epsilon}_1\bar{\sigma}^0\psi^e\lambda^f\lambda^g \\
&+i\sqrt{2}\mathcal{I}^{ed}\epsilon^{efg}A^{\dagger g}\bar{\epsilon}_1\bar{\sigma}^0\lambda^d\epsilon_1\psi^f
\end{aligned}
\tag{F.35}
$$

$$
\begin{aligned}
\text{terms III} \;=\; &-\frac{i}{2}(\mathcal{I}^{ae})^{-1}\mathcal{F}^{\dagger ab}\Pi^{ej}v^{*oib}\epsilon_1\sigma_i\bar{\sigma}^0\sigma_j\bar{\epsilon}_1 \\
&-\int d^3xd^3y[\mathcal{F}^{\dagger ab}(x)\epsilon_1\sigma_i\bar{\lambda}^a(x)\bar{\epsilon}_1\bar{\sigma}_j\lambda^e(y)\epsilon^{0jik} \\
&\qquad\times(\delta^{eb}\partial_k^x+\epsilon^{ebh}v_k^h(x))\delta^{(3)}(\vec{x}-\vec{y})] \\
&+\frac{i}{\sqrt{2}}\mathcal{F}^{abe\dagger}\bar{\epsilon}_1\bar{\psi}^e\epsilon_1\sigma_i\bar{\lambda}^a v^{*0ib} \\
&-\frac{1}{2\sqrt{2}}(\mathcal{I}^{ag})^{-1}\mathcal{F}^{ab\dagger}\bar{\epsilon}_1\bar{\psi}^e\epsilon_1\sigma_i\bar{\lambda}^f v^{*0ib}\mathcal{F}^{efg\dagger} \\
&-\frac{i}{4}(\mathcal{I}^{ae})^{-1}\mathcal{F}^{\dagger ab}\mathcal{F}^{ef}v^{*0ib}v^{*0jf}\epsilon_1\sigma_i\bar{\sigma}^0\sigma_j\bar{\epsilon}_1 \\
&+\frac{1}{2\sqrt{2}}(\mathcal{I}^{af})^{-1}\mathcal{F}^{\dagger ab}\mathcal{F}^{efg}v^{*0ib}\bar{\epsilon}_1\bar{\sigma}^0\psi^e\epsilon_1\sigma_i\bar{\sigma}^0\lambda^g \\
&-\frac{i}{2}\mathcal{F}^{\dagger ab}\epsilon^{efg}A^fA^{\dagger g}v^{*0ib}\bar{\epsilon}_1\bar{\sigma}_i\epsilon_1
\end{aligned}
\tag{F.36}
$$

$$
\text{terms IV} \;=\; \frac{1}{\sqrt{2}}(\mathcal{I}^{ec})^{-1}\mathcal{F}^{\dagger abc}\Pi^{ej}\epsilon_1\sigma^0\bar{\psi}^a\bar{\epsilon}_1\bar{\sigma}_j\sigma^0\bar{\lambda}^b
$$

$$
\begin{aligned}
&-\frac{1}{2}\mathcal{F}^{\dagger abce}\bar{\epsilon}_1\bar{\psi}^e\epsilon_1\sigma^0\bar{\psi}^a\bar{\lambda}^b\bar{\lambda}^c \\
&-\frac{i}{4}\mathcal{F}^{\dagger abc}\mathcal{F}^{\dagger efg}(\mathcal{I}^{ag})^{-1}\bar{\epsilon}_1\bar{\psi}^e\bar{\lambda}^b\bar{\lambda}^c\epsilon_1\sigma^0\bar{\psi}^f \\
&-\frac{i}{2}\mathcal{F}^{\dagger abc}\mathcal{F}^{\dagger efg}(\mathcal{I}^{gc})^{-1}\bar{\epsilon}_1\bar{\psi}^e\epsilon_1\sigma^0\bar{\psi}^a\bar{\lambda}^f\bar{\lambda}^b \\
&+\frac{1}{2\sqrt{2}}\mathcal{F}^{\dagger abc}\mathcal{F}^{ef}(\mathcal{I}^{ec})^{-1}v^{*0jf}\epsilon_1\sigma^0\bar{\psi}^a\bar{\epsilon}_1\bar{\sigma}_j\sigma^0\bar{\lambda}^b \\
&-\frac{i}{8}\mathcal{F}^{\dagger abc}\mathcal{F}^{efg}(\mathcal{I}^{ae})^{-1}\bar{\lambda}^b\bar{\lambda}^c\lambda^f\lambda^g\epsilon_1\sigma^0\bar{\epsilon}_1 \\
&+\frac{i}{2}\mathcal{F}^{\dagger abc}\mathcal{F}^{efg}(\mathcal{I}^{cf})^{-1}\bar{\lambda}^b\bar{\sigma}^0\lambda^g\epsilon_1\sigma^0\bar{\psi}^a\bar{\epsilon}_1\bar{\sigma}^0\psi^e \\
&-\frac{1}{\sqrt{2}}\mathcal{F}^{\dagger abc}\epsilon_1\sigma^0\bar{\psi}^a\bar{\epsilon}_1\bar{\lambda}^c\epsilon^{bfg}A^fA^{\dagger g}
\end{aligned}
\tag{F.37}
$$

$$
\begin{aligned}
\text{terms V} \;=\; &-i\epsilon^{acd}\mathcal{I}^{ab}(\mathcal{I}^{be})^{-1}\epsilon_1\sigma_j\bar{\epsilon}_1\Pi^{ej}A^cA^{\dagger d} \\
&-\frac{1}{\sqrt{2}}\epsilon^{acd}\mathcal{F}^{abe\dagger}\bar{\epsilon}_1\bar{\psi}^e\epsilon_1\sigma^0\bar{\lambda}^bA^cA^{\dagger d} \\
&+i\sqrt{2}\epsilon^{acd}\epsilon_1\sigma^0\bar{\lambda}^b\bar{\epsilon}_1\bar{\psi}^dA^c\mathcal{I}^{ab} \\
&-\frac{1}{\sqrt{2}}\epsilon^{acd}\mathcal{I}^{ab}(\mathcal{I}^{bg})^{-1}\mathcal{F}^{\dagger efg}A^cA^{\dagger d}\bar{\epsilon}_1\bar{\psi}^e\epsilon_1\sigma^0\bar{\lambda}^f \\
&-i\frac{1}{2}\epsilon^{acd}\mathcal{I}^{ab}(\mathcal{I}^{be})^{-1}\mathcal{F}^{ef}A^cA^{\dagger d}v^{*0jf}\epsilon_1\sigma_j\bar{\epsilon}_1 \\
&+\frac{1}{\sqrt{2}}\epsilon^{acd}\mathcal{I}^{ab}(\mathcal{I}^{bf})^{-1}\mathcal{F}^{efg}A^cA^{\dagger d}\bar{\epsilon}_1\bar{\sigma}^0\psi^e\epsilon_1\lambda^g \\
&+i\epsilon^{acd}\epsilon^{bfg}\mathcal{I}^{ab}A^cA^{\dagger d}A^fA^{\dagger g}\epsilon_1\sigma^0\bar{\epsilon}_1
\end{aligned}
\tag{F.38}
$$

$$
\begin{aligned}
\text{terms VI} \;=\; &\frac{1}{\sqrt{2}}(\mathcal{I}^{ec})^{-1}\mathcal{F}^{cad}\epsilon_1\psi^a\Pi^{ej}\bar{\epsilon}_1\bar{\sigma}_j\lambda^d \\
&-\frac{i}{2}(\mathcal{I}^{gc})^{-1}\mathcal{F}^{cad}\mathcal{F}^{\dagger efg}\bar{\epsilon}_1\bar{\psi}^e\epsilon_1\psi^a[\bar{\psi}^f\bar{\sigma}^0\psi^d+\bar{\lambda}^f\bar{\sigma}^0\lambda^d] \\
&+\frac{1}{2\sqrt{2}}(\mathcal{I}^{ec})^{-1}\mathcal{F}^{cad}\mathcal{F}^{ef}v^{*0jf}\epsilon_1\psi^a\bar{\epsilon}_1\bar{\sigma}_j\lambda^d \\
&-\frac{i}{4}(\mathcal{I}^{ec})^{-1}\mathcal{F}^{cad}\mathcal{F}^{efg}\epsilon_1\psi^a\bar{\epsilon}_1\bar{\sigma}^0\psi^d\lambda^f\lambda^g \\
&-\frac{i}{2}(\mathcal{I}^{ec})^{-1}\mathcal{F}^{cad}\mathcal{F}^{efg}\epsilon_1\psi^a\bar{\epsilon}_1\bar{\sigma}^0\psi^f\lambda^d\lambda^g]
\end{aligned}
\tag{F.39}
$$

We now get rid of ϵ_1 and $\bar{\epsilon}_1$ by using the property $\{\epsilon_1 Q_1\ ,\ \bar{\epsilon}_1\bar{Q}_1\}_- = \epsilon_1^\alpha\bar{\epsilon}_1^{\dot{\alpha}}\{Q_{1\alpha}\ ,\ \bar{Q}_{1\dot{\alpha}}\}_+$ and after that we take the trace with $\bar{\sigma}^{0\dot{\alpha}\alpha}$ $(\bar{\sigma}^0 = -\bar{\sigma}_0)$.

The non zero terms are all collected in the following. Note that we have still to divide by a factor $4i$ and note also that, for instance, II(6) stands for the sixth term in the group II and so forth.

"Classical" terms

Kinetic terms for the e.m. field[1]

$$\mathrm{I}(1) \qquad -2i(\mathcal{I}^{ab})^{-1}\Pi^{ai}\Pi^{bi} \tag{F.40}$$

$$\mathrm{I}(5)+\mathrm{III}(1) \qquad -2i(\mathcal{I}^{ab})^{-1}\mathcal{R}^{bc}\Pi^{ai}v^{*0ic} \tag{F.41}$$

$$\mathrm{III}(5) \qquad -\frac{i}{2}(\mathcal{I}^{ae})^{-1}\mathcal{F}^{\dagger ab}\mathcal{F}^{ef}v^{*0ib}v^{*0if} \tag{F.42}$$

Kinetic terms for the scalar fields

$$\begin{aligned}\mathrm{II}(3)+\mathrm{II}(4) \qquad & -4i(\mathcal{I}^{ab})^{-1}\pi^a_A[\pi^b_{A^\dagger}+\frac{1}{2}\mathcal{F}^{\dagger bcd}(\bar\psi^c\bar\sigma^0\psi^d+\bar\lambda^c\bar\sigma^0\lambda^d)] \\ & = -4i(\mathcal{I}^{ab})^{-1}\pi^a_A(\pi^b_A)^\dagger\end{aligned} \tag{F.43}$$

$$\mathrm{II}(6) \qquad 4i\mathcal{I}^{ab}(\mathcal{D}^iA^a)(\mathcal{D}^iA^{\dagger b}) \tag{F.44}$$

Kinetic terms for the spinors

for λ:

$$\mathrm{I}(4) \qquad 2i\mathcal{F}^{ab}\lambda^a\sigma^i\mathcal{D}_i\bar\lambda^b \tag{F.45}$$

$$\mathrm{III}(2) \qquad -2i\mathcal{F}^{\dagger ab}\bar\lambda^a\bar\sigma^i\mathcal{D}_i\lambda^b \tag{F.46}$$

for ψ:

$$\mathrm{II}(2) \qquad -2\mathcal{I}^{ab}\psi^a\sigma^i\mathcal{D}_i\bar\psi^b \tag{F.47}$$

$$\mathrm{II}(7) \qquad -2\mathcal{I}^{ab}\bar\psi^a\bar\sigma^i\mathcal{D}_i\psi^b \tag{F.48}$$

$$\mathrm{II}(4)+\mathrm{III}(5) \qquad 2\bar\psi^e\bar\sigma^i\psi^g[\partial_i\mathcal{I}^{ab}+\epsilon^{ebc}v^b_i\mathcal{I}^{gc}+\epsilon^{gbc}v^b_i\mathcal{I}^{ec}] \tag{F.49}$$

[1] Classical test on the e.m. kinetic piece: $\mathcal{I}^{-1}\Pi^2+\mathcal{I}^{-1}\mathcal{R}\Pi v^*+\frac{1}{4}\mathcal{I}^{-1}(\mathcal{R}^2+\mathcal{I}^2)v^{*2}=\mathcal{I}(E^2+B^2)$ where $v^*=2B$ and $\Pi=-(\mathcal{I}E+\mathcal{R}B)$.

to write the last term (purely quantum) we used the properties given in the first Section of this Appendix. Integrating by parts we have

$$-4\mathcal{I}^{ab}\lambda^a\sigma^i\mathcal{D}_i\bar{\lambda}^b + 2i(\partial\mathcal{F}^{\dagger ab})\bar{\lambda}^a\bar{\sigma}^i\lambda^b + \partial_i(-2i\mathcal{F}^{\dagger ab}\bar{\lambda}^a\bar{\sigma}^i\lambda^b) \qquad \text{(F.50)}$$

$$-4\mathcal{I}^{ab}\psi^a\sigma^i\mathcal{D}_i\bar{\psi}^b + \partial_i(-2\mathcal{I}^{ab}\bar{\psi}^a\bar{\sigma}^i\psi^b) \qquad \text{(F.51)}$$

Yukawa potential

$$\mathrm{I}(3) \qquad -i3\sqrt{2}\epsilon^{eah}\mathcal{I}^{ef}A^h\bar{\psi}^f\bar{\lambda}^a \qquad \text{(F.52)}$$

$$\mathrm{V}(3) \qquad -i\sqrt{2}\epsilon^{acd}\mathcal{I}^{ad}A^c\bar{\psi}^d\bar{\lambda}^b \qquad \text{(F.53)}$$

$$\mathrm{IV}(8)+\mathrm{V}(2)+\mathrm{V}(4) \qquad -\frac{3}{\sqrt{2}}\epsilon^{bfg}\mathcal{F}^{\dagger abc}A^fA^{\dagger g}\bar{\psi}^a\bar{\lambda}^c \qquad \text{(F.54)}$$

$$\mathrm{II}(1) \qquad -3i\sqrt{2}\epsilon^{aeh}\mathcal{I}^{ab}A^{\dagger h}\psi^b\lambda^e \qquad \text{(F.55)}$$

$$\mathrm{II}(10) \qquad -i\sqrt{2}\epsilon^{efg}\mathcal{I}^{ed}A^{\dagger g}\psi^f\lambda^d \qquad \text{(F.56)}$$

$$\mathrm{V}(6) \qquad -\frac{1}{\sqrt{2}}\epsilon^{acd}\mathcal{F}^{eag}A^cA^{\dagger d}\psi^e\lambda^g \qquad \text{(F.57)}$$

By using the properties of $\epsilon^{abc}\mathcal{F}^{ade}$ listed in the previous Section of this Appendix we can recast these terms into

$$-i2\sqrt{2}\epsilon^{abc}\mathcal{I}^{ad}(A^c\bar{\psi}^d\bar{\lambda}^b + A^{\dagger c}\psi^d\lambda^b) \qquad \text{(F.58)}$$

Higgs potential

$$\mathrm{V}(7) \quad 2i\epsilon^{acd}\mathcal{I}^{ab}\epsilon^{bfg}A^cA^{\dagger d}A^fA^{\dagger g} \qquad \text{(F.59)}$$

Purely quantum corrections

Terms that will contribute to $\mathcal{F}^{abc}\sigma_{\mu\nu}v^{\mu\nu}$

and to dummy fields on shell via the two fermions piece Π_{F}

$$\mathrm{I}(2)+\mathrm{IV}(1) \qquad 2\sqrt{2}(\mathcal{I}^{ec})^{-1}\mathcal{F}^{\dagger abc}\Pi^{ei}\bar{\psi}^a\bar{\sigma}_{i0}\bar{\lambda}^b \qquad \text{(F.60)}$$

$$\mathrm{I}(6)+\mathrm{VI}(1) \qquad -2\sqrt{2}(\mathcal{I}^{af})^{-1}\mathcal{F}^{feg}\Pi^{ai}\psi^{e}\sigma_{i0}\lambda^{g} \tag{F.61}$$

$$\mathrm{III}(4)+\mathrm{IV}(5) \qquad \sqrt{2}(\mathcal{I}^{ag})^{-1}\mathcal{R}^{ab}\mathcal{F}^{\dagger efg}v^{*0ib}\bar{\psi}^{e}\bar{\sigma}_{i0}\bar{\lambda}^{f} \tag{F.62}$$

$$\mathrm{III}(6)+\mathrm{VI}(3) \qquad -\sqrt{2}(\mathcal{I}^{af})^{-1}\mathcal{R}^{ab}\mathcal{F}^{efg}v^{*0ib}\psi^{e}\sigma_{i0}\lambda^{g} \tag{F.63}$$

$$\mathrm{III}(3) \qquad i\sqrt{2}\mathcal{F}^{\dagger abc}v^{*0ia}\bar{\psi}^{b}\bar{\sigma}_{i0}\bar{\lambda}^{c} \tag{F.64}$$

$$\mathrm{II}(8) \qquad i\sqrt{2}\mathcal{F}^{abc}v^{*0ia}\psi^{b}\sigma_{i0}\lambda^{c} \tag{F.65}$$

Summing them up we obtain[2]

$$-2\sqrt{2}(\mathcal{I}^{af})^{-1}\mathcal{F}^{feg}\psi^{e}\sigma_{i0}\lambda^{g}(\Pi^{ia}+\mathcal{F}^{\dagger ab}B^{ib}) \tag{F.66}$$

$$2\sqrt{2}(\mathcal{I}^{ec})^{-1}\mathcal{F}^{\dagger abc}\bar{\psi}^{a}\bar{\sigma}_{i0}\bar{\lambda}^{b}(\Pi^{ie}+\mathcal{F}^{ed}B^{id}) \tag{F.67}$$

Terms that contribute to the dummy fields on-shell only

$$\mathrm{VI}(2) \qquad \frac{i}{4}\mathcal{F}^{\dagger efg}\mathcal{F}^{cad}(\mathcal{I}^{gc})^{-1}\bar{\psi}^{e}\bar{\psi}^{f}\psi^{a}\psi^{c} \tag{F.68}$$

$$\mathrm{VI}(4) \qquad \frac{3i}{4}\mathcal{F}^{bec}\mathcal{F}^{efg}(\mathcal{I}^{ab})^{-1}\psi^{a}\psi^{c}\lambda^{f}\lambda^{g} \tag{F.69}$$

$$\mathrm{IV}(3)+\mathrm{VII}(4) \qquad \frac{3i}{4}\mathcal{F}^{\dagger bec}\mathcal{F}^{\dagger efg}(\mathcal{I}^{ab})^{-1}\bar{\psi}^{a}\bar{\psi}^{c}\bar{\lambda}^{f}\bar{\lambda}^{g} \tag{F.70}$$

$$\mathrm{IV}(6) \qquad +\frac{i}{4}\mathcal{F}^{\dagger abc}\mathcal{F}^{efg}(\mathcal{I}^{ae})^{-1}\bar{\lambda}^{b}\bar{\lambda}^{c}\lambda^{f}\lambda^{g} \tag{F.71}$$

$\mathcal{F}^{abcd}$*-type of terms*

$$\mathrm{II}(9)+\mathrm{IV}(2) \qquad -\frac{1}{2}(\mathcal{F}^{abcd}\psi^{a}\psi^{b}\lambda^{c}\lambda^{d}-\mathcal{F}^{\dagger abcd}\bar{\psi}^{a}\bar{\psi}^{b}\bar{\lambda}^{c}\bar{\lambda}^{d}) \tag{F.72}$$

Collecting all these terms and dividing by $4i$ we obtain the Hamiltonian given in Chapter 4.

[2] $\Pi+\frac{1}{2}\mathcal{F}''v^{*}=-\frac{1}{2i}\mathcal{I}\hat{v}^{\dagger}+\Pi_{\mathrm{F}}$ and $\Pi+\frac{1}{2}\mathcal{F}^{\dagger\prime\prime}v^{*}=-\frac{1}{2i}\mathcal{I}\hat{v}+\Pi_{\mathrm{F}}$. Also $\hat{v}=E+iB$ and $\hat{v}^{\dagger}=E-iB$.

F.3 Tests on the Lagrangian

Let us first rewrite the four fermions terms in the SU(2) Lagrangian

$$\begin{aligned}
\mathcal{L}_{\mathrm{F}}^{\mathrm{SU(2)}} &= \frac{3}{16}(\mathcal{I}^{ab})^{-1}\Big[\mathcal{F}^{acd}\mathcal{F}^{bef}(\psi^d\psi^f\lambda^c\lambda^e - \psi^d\lambda^e\lambda^c\psi^f) - \mathcal{F}^{gbc}\mathcal{F}^{gef}\psi^a\psi^c\lambda^e\lambda^f \\
&+ \mathcal{F}^{\dagger acd}\mathcal{F}^{\dagger bef}(\bar\psi^d\bar\psi^f\bar\lambda^c\bar\lambda^e - \bar\psi^d\bar\lambda^e\bar\lambda^c\bar\psi^f) - \mathcal{F}^{\dagger gbc}\mathcal{F}^{\dagger gef}\bar\psi^a\bar\psi^c\bar\lambda^e\bar\lambda^f \\
&+ \mathcal{F}^{\dagger acd}\mathcal{F}^{\dagger bef}(\lambda^c\sigma^0\bar\psi^f\psi^d\sigma^0\bar\lambda^e - \lambda^c\sigma^0\bar\lambda^e\psi^d\sigma^0\bar\psi^f)\Big] \\
&- \frac{1}{16}(\mathcal{I}^{ab})^{-1}\mathcal{F}^{\dagger acd}\mathcal{F}^{bef}(\bar\psi^c\bar\psi^d\psi^e\psi^f + \bar\lambda^c\bar\lambda^d\lambda^e\lambda^f) \\
&+ \frac{1}{2i}(\frac{1}{4}\mathcal{F}^{abcd}\psi^a\psi^b\lambda^c\lambda^d - \frac{1}{4}\mathcal{F}^{\dagger abcd}\bar\psi^a\bar\psi^b\bar\lambda^c\bar\lambda^d)
\end{aligned} \tag{F.73}$$

We now want to compare these terms with the U(1) correspondent ones. At this end we write here the four fermions contributions to the U(1) effective Lagrangian obtained after elimination of the dummy fields

$$\begin{aligned}
\mathcal{L}_{\mathrm{F}}^{\mathrm{U(1)}} &= \frac{3}{32}\mathcal{I}^{-1}(\mathcal{F}''')^2\psi^2\lambda^2 + \frac{3}{32}\mathcal{I}^{-1}(\mathcal{F}^{\dagger'''})^2\bar\psi^2\bar\lambda^2 \\
&- \frac{1}{16}\mathcal{I}^{-1}\mathcal{F}^{\dagger'''}\mathcal{F}'''(\bar\psi^2\psi^2 + \bar\lambda^2\lambda^2 + 2\psi\lambda\bar\psi\bar\lambda) \\
&+ \frac{1}{2i}(\frac{1}{4}\mathcal{F}''''\psi^2\lambda^2 - \frac{1}{4}\mathcal{F}^{\dagger''''}\bar\psi^2\bar\lambda^2)
\end{aligned} \tag{F.74}$$

We see immediately that the $\mathcal{F}''''$ terms, the $\mathcal{F}^{\dagger'''}\mathcal{F}'''\psi^2\bar\psi^2$ and $\mathcal{F}^{\dagger'''}\mathcal{F}'''\lambda^2\bar\lambda^2$ have the correct factors. For the $(\mathcal{F}''')^2\psi^2\lambda^2$ terms we have simply to notice that in the Abelian limit

$$\frac{3}{16}(\mathcal{I}^{ab})^{-1}\mathcal{F}^{acd}\mathcal{F}^{bef}(\psi^d\psi^f\lambda^c\lambda^e - \psi^d\lambda^e\lambda^c\psi^f) - \mathcal{F}^{gbc}\mathcal{F}^{gef}\psi^a\psi^c\lambda^e\lambda^f \tag{F.75}$$

reduces to

$$-\frac{3}{16}\mathcal{I}^{-1}(\mathcal{F}''')^2\psi\lambda\lambda\psi \tag{F.76}$$

and by using the Fierz identities given in Appendix B, $\psi\lambda\lambda\psi = -\frac{1}{2}\psi^2\lambda^2$, we obtain the correct factor $\frac{3}{32}$. Similarly for the $(\mathcal{F}^{\dagger'''})^2\bar\psi^2\bar\lambda^2$ terms. For the $\mathcal{F}^{\dagger'''}\mathcal{F}'''\psi\lambda\bar\psi\lambda$ terms we cannot recast them into a proper form. Nevertheless

there is no other terms to combine them with and we conclude that they must give the right factors.

Note also that

$$
\begin{aligned}
\mathcal{L}^{\mathrm{SU}(2)}_{\mathrm{e.m.}} &= \frac{1}{2i}\Big[-\frac{1}{4}\mathcal{F}^{ab}v^{a\mu\nu}\hat{v}^{b}_{\mu\nu}+\frac{1}{4}\mathcal{F}^{\dagger ab}v^{a\mu\nu}\hat{v}^{\dagger b}_{\mu\nu}\Big] \\
&= \frac{1}{2i}\Big[-\frac{1}{4}\mathcal{F}^{ab}(2v^{a0i}v^{b}_{0i}+iv^{a0i}v^{*b}_{0i}+v^{aij}v^{b}_{ij}+\frac{i}{2}v^{aij}v^{*b}_{ij}) \\
&\quad +\frac{1}{4}\mathcal{F}^{\dagger ab}(2v^{a0i}v^{b}_{0i}-iv^{a0i}v^{*b}_{0i}+v^{aij}v^{b}_{ij}-\frac{i}{2}v^{aij}v^{*b}_{ij})\Big] \\
&= -\frac{1}{2}\mathcal{I}^{ab}E^{ai}E^{bi}-\frac{1}{2}\mathcal{R}^{ab}E^{ai}B^{bi}+\frac{1}{2}\mathcal{I}^{ab}B^{ai}B^{bi}-\frac{1}{2}\mathcal{R}^{ab}E^{ai}B^{bi}
\end{aligned}
\tag{F.77}
$$

The term $\mathcal{R}EB$ is the effective version of the CP violating θ term.

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